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NUMERICAL NON-SIMULTANEOUS BLOW-UP FOR A REACTION-DIFFUSION SYSTEM WITH THREE COMPONENTS

EDJA KOUAMÉ BÉRANGER [1], N'GUESSAN YAO MATHIEU [2], N'GUESSAN KOFFI [3] AND TOURÉ KIDJÉGBO AUGUSTIN [4]

Abstract. This paper deals with the study of a numerical approximation of a reaction-diffusion system with three components. Under some assumptions, we prove that the solution of a semidiscrete form of above problem blows up in a finite time and estimate its semidiscrete blow-up time. We show that the semidiscrete non-simultaneous blow-up occurs. We also establish that the semidiscrete blow-up time converges to the theoretical one as the mesh size tends to zero. Finally, we give some numerical experiments to illustrate our analysis.

1. Introduction

Consider the following nonlinear parabolic system

$$
\begin{aligned}
u_t(x,t) &= u_{xx}(x,t),\ (x,t)\in(0,1)\times(0,T),\\
v_t(x,t) &= v_{xx}(x,t),\ (x,t)\in(0,1)\times(0,T),\\
w_t(x,t) &= w_{xx}(x,t),\ (x,t)\in(0,1)\times(0,T),\\
-u_x(0,t) &= u^{p_{11}}(0,t)+v^{p_{12}}(0,t),\ t\in(0,T),\\
-v_x(0,t) &= v^{p_{22}}(0,t)+w^{p_{23}}(0,t),\ t\in(0,T),\\
-w_x(0,t) &= w^{p_{33}}(0,t)+u^{p_{31}}(0,t),\ t\in(0,T),\\
u_x(1,t) &= u^{p_{11}}(1,t)+v^{p_{12}}(1,t),\ t\in(0,T),\\
v_x(1,t) &= v^{p_{22}}(1,t)+w^{p_{23}}(1,t),\ t\in(0,T),\\
w_x(1,t) &= w^{p_{33}}(1,t)+u^{p_{31}}(1,t),\ t\in(0,T),\\
u(x,0) = u_0(x), v(x,0) &= v_0(x), w(x,0)=w_0(x),\ x\in(0,1),
\end{aligned}
\tag{1.1}
$$

where p_{11}, p_{22}, p_{33}, p_{12}, p_{23}, $p_{31}\geq 0$; $u_0(x)$, $v_0(x)$ and $w_0(x)$ are positive smooth functions satisfying the compatibility conditions.

Problem (1.1) comes from chemical reactions, heat transfer, where $u_0(x)$, $v_0(x)$ and $w_0(x)$ represent the thickness of three kinds of chemical reactants, the temperatures of three different materials during a propagation. Phenomena of blow-up and non-simultaneous blow-up for nonlinear parabolic systems have been studied by several authors (see [2]-[6]). In [3] the authors have shown that only one component of the solution (u,v,w) of (1.1) blows up in a finite time $T>0$. If ($p_{22}\leq 1$, $p_{33}\leq 1$ and $p_{31}+1<p_{11}$), then only u blows up for every positive initial data. Also, if ($p_{11}\leq 1$,

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$p_{33} \leq 1$ and $p_{12}+1 < p_{22}$), then only v blows up for every positive initial data and if ($p_{11} \leq 1$, $p_{22} \leq 1$ and $p_{23}+1 < p_{33}$), then only w blows up for every positive initial data.

They have also proved that only one component of the solution (u, v, w) of (1.1) remains bounded. If $1 < p_{22} < p_{12}+1$, $p_{11} \leq 1$, $p_{33} \leq 1$, and $p_{31} < \dfrac{p_{22}-1}{p_{12}+1-p_{22}}$, then, for every initial data, u and v blow up simultaneously at T while w remains bounded up to time T. If $1 < p_{11} < p_{31}+1$, $p_{22} \leq 1$, $p_{33} \leq 1$, and $p_{23} < \dfrac{p_{11}-1}{p_{31}+1-p_{11}}$, then, for every initial data, u and w blow up simultaneously at T while v remains bounded up to time T. Finally, if $1 < p_{33} < p_{23}+1$, $p_{11} \leq 1$, $p_{22} \leq 1$, and $p_{12} < \dfrac{p_{33}-1}{p_{23}+1-p_{33}}$, then, for every initial data, v and w blow up simultaneously at T while u remains bounded up to time T.

We say that the solution (u, v, w) of (1.1) blows up non-simultaneously in a finite time $T > 0$ when one or two of its three components blow up at T while the others remain bounded on $[0, T)$.

But these authors do not provide us with exact value of the non-simultaneous blow-up time of the solution of problem (1.1)although necessary for the work of some engineers. This is why we opt for a numerical study of this problem to determine good approximate values of the non-simultaneous blow-up time of its solution. In this paper, we are interesting in the numerical study of the above problem. A similar study has been undertaken in ([1]).

Let $I \geqslant 3$ be a positive integer and let $h = \frac{1}{I-1}$. Define the grid $x_i = (i-1)h$ with $i = 1, \ldots, I$. Approximate the solution (u, v, w) of (1.1) by the solution $\big(U_h(t) = (U_1(t), \ldots, U_I(t))^T, V_h(t) = (V_1(t), \ldots, V_I(t))^T, W_h(t) = (W_1(t), \ldots, W_I(t))^T\big)$ and approximate the initial data (u_0, v_0, w_0) of the same problem by $\big(\psi_{1,h} = (\psi_{1,1}, \ldots, \psi_{1,I})^T, \psi_{2,h} = (\psi_{2,1}, \ldots, \psi_{2,I})^T, \psi_{3,h} = (\psi_{3,1}, \ldots, \psi_{3,I})^T\big)$ of the following system of ODEs which is obtained using the finite difference method.

$$
\begin{aligned}
U_i'(t) &= \delta^2 U_i(t) + a_i\left(U_i^{p_{11}}(t) + V_i^{p_{12}}(t)\right), \quad i = 1, \ldots, I,\ t \in (0, T_h), && (1.2)\\
V_i'(t) &= \delta^2 V_i(t) + a_i\left(V_i^{p_{22}}(t) + W_i^{p_{23}}(t)\right), \quad i = 1, \ldots, I,\ t \in (0, T_h), && (1.3)\\
W_i'(t) &= \delta^2 W_i(t) + a_i\left(W_i^{p_{33}}(t) + U_i^{p_{31}}(t)\right), \quad i = 1, \ldots, I,\ t \in (0, T_h), && (1.4)\\
U_i(0) &= \psi_{1,i}, \quad V_i(0) = \psi_{2,i}, \quad W_i(0) = \psi_{3,i}, \qquad i = 1, \ldots, I, && (1.5)
\end{aligned}
$$

where

$$
p_{11},\ p_{22},\ p_{33},\ p_{12},\ p_{23},\ p_{31} \geq 0, \quad \psi_{1,i},\ \psi_{2,i},\ \psi_{3,i} > 0, \quad i = 1, \ldots, I,
$$

$$
\delta^2 U_i(t) = \frac{U_{i-1}(t) - 2U_i(t) + U_{i+1}(t)}{h^2},\ 2 \leqslant i \leqslant I-1,\ t \in (0, T_h),
$$

$$
\delta^2 U_1(t) = \frac{2U_2(t) - 2U_1(t)}{h^2},\ \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2},\ t \in (0, T_h),
$$

$$
a_1 = \frac{2}{h},\ a_I = \frac{2}{h},\ \text{and } a_i = 0,\ i = 2, \ldots, I-1.
$$

The paper is organized as follows. In section 1, using the finite difference method we construct a semidiscrete scheme of the continuous problem. In section 2, we give some properties concerning the semidiscrete scheme. In Section 3, under some conditions, we prove that the solution of a semidiscrete form of the continuous problem blows up in a finite time and estimate its semidiscrete blow-up time. In section 4, we propose the criteria of the non-simultaneous blow-up of the semidiscrete scheme. In Section 5, we show the convergence of the semidiscrete blow-up time to the theoretical one when the mesh size goes to zero. Finally, in the last section, we give some numerical experiments for a best illustration of our analysis.

2. Properties of the semidiscrete scheme

In this section, we give some properties of the problem (1.2)–(1.5).

Definition 2.1. We say that $(\underline{U_h},\underline{V_h},\underline{W_h})\in\big(C^1([0,T_h),\mathbb{R}^I)\big)^3$ is a lower solution of (1.2)–(1.5) if

$$\begin{aligned}
\underline{U_i}'(t) &\leqslant \delta^2\underline{U_i}(t)+a_i(\underline{U_i}^{p_{11}}(t)+\underline{V_i}^{p_{12}}(t)), \quad i=1,\dots,I,\ t\in(0,T_h),\\
\underline{V_i}'(t) &\leqslant \delta^2\underline{V_i}(t)+a_i(\underline{V_i}^{p_{22}}(t)+\underline{W_i}^{p_{23}}(t)), \quad i=1,\dots,I,\ t\in(0,T_h),\\
\underline{W_i}'(t) &\leqslant \delta^2\underline{W_i}(t)+a_i(\underline{W_i}^{p_{33}}(t)+\underline{U_i}^{p_{31}}(t)), \quad i=1,\dots,I,\ t\in(0,T_h),\\
0<\underline{U_i}(0)\leqslant\psi_{1,i},\ 0<\underline{V_i}(0)&\leqslant\psi_{2,i},\ 0<\underline{W_i}(0)\leqslant\psi_{3,i} \qquad i=1,\dots,I,
\end{aligned}$$

where (U_h,V_h,W_h) is the solution of (1.2)–(1.5). On the other hand, we say that $(\overline{U_h},\overline{V_h},\overline{W_h})\in\big(C^1([0,T_h),\mathbb{R}^I)\big)^3$ is an upper solution of (1.2)–(1.5) if these inequalities are reversed.

The following lemma is the discrete form of the maximum principle.

Lemma 2.1. *Let $c_h,b_h,\theta_h,\kappa_h,\mu_h,\rho_h\in C^0([0,T_h),\mathbb{R}^I)$ and $U_h,V_h,W_h\in C^1([0,T_h),\mathbb{R}^I)$ such that*

$$\begin{aligned}
U_i'(t)-\delta^2U_i(t)-c_i(t)U_i(t)-b_i(t)V_i(t)&\geqslant0, \quad i=1,\dots,I,\ t\in(0,T_h),\\
V_i'(t)-\delta^2V_i(t)-\theta_i(t)U_i(t)-\kappa_i(t)V_i(t)&\geqslant0, \quad i=1,\dots,I,\ t\in(0,T_h),\\
W_i'(t)-\delta^2W_i(t)-\mu_i(t)W_i(t)-\rho_i(t)V_i(t)&\geqslant0, \quad i=1,\dots,I,\ t\in(0,T_h),\\
U_i(0)\geqslant0,\ V_i(0)\geqslant0,\ W_i(0)&\geqslant0, \qquad i=1,\dots,I.
\end{aligned}$$

Then we have

$$U_i(t)\geqslant0,\ V_i(t)\geqslant0,\ W_i(t)\geqslant0, \quad i=1,\dots,I,\ t\in(0,T_h).$$

Proof. Let $T_0<T_h$ and let $(H_h(t),L_h(t),J_h(t))=(\mathrm{e}^{\xi t}U_h(t),\mathrm{e}^{\xi t}V_h(t),\mathrm{e}^{\xi t}W_h(t))$ where ξ is a real. We find that $(H_h(t),L_h(t),J_h(t))$ satisfies the following inequalities :

$$H_i'(t)-\delta^2H_i(t)-(c_i(t)+\xi)H_i(t)-b_i(t)L_i(t)\geqslant0, \quad i=1,\dots,I,\ t\in(0,T_h), \tag{2.1}$$

$$L_i'(t)-\delta^2L_i(t)-(\theta_i(t)+\xi)L_i(t)-\kappa_i(t)J_i(t)\geqslant0, \quad i=1,\dots,I,\ t\in(0,T_h), \tag{2.2}$$

$$J_i'(t)-\delta^2J_i(t)-(\mu_i(t)+\xi)J_i(t)-\rho_i(t)H_i(t)\geqslant0, \quad i=1,\dots,I,\ t\in(0,T_h), \tag{2.3}$$

$$H_i(0)\geqslant0,\ L_i(0)\geqslant0,\ J_i(0)\geqslant0, \quad i=1,\dots,I. \tag{2.4}$$

Set $m = \min\left\{\min_{1\leqslant i\leqslant I, t\in[0,T_0]} H_i(t), \min_{1\leqslant i\leqslant I, t\in[0,T_0]} L_i(t), \min_{1\leqslant i\leqslant I, t\in[0,T_0]} J_i(t)\right\}$. Since for $i = 1, \ldots, I$, H_i, L_i and J_i are continuous functions on a compact, we assume that $m = H_{i_0}(t_{i_0})$ for a certain $i_0 = 1, \ldots, I$.
Assume $m < 0$ and $\xi < 0$ such that

$$(c_{i_0}(t_{i_0}) + \xi) < 0, \quad (\theta_{i_0}(t_{i_0}) + \xi) < 0 \quad \textit{and} \quad (\mu_{i_0}(t_{i_0}) + \xi) < 0.$$

If $t_{i_0} = 0$, then $H_{i_0}(0) < 0$, which contradicts (2.4), hence $t_{i_0} \neq 0$.
if $1 \leqslant i_0 \leqslant I$, we have

$$H'_{i_0}(t_{i_0}) = \lim_{k\to 0} \frac{H_{i_0}(t_{i_0}) - H_{i_0}(t_{i_0} - k)}{k} \leqslant 0 \quad \textit{and} \quad \delta^2 H_{i_0}(t_{i_0}) \geq 0.$$

With a straightforward computation we get

$$H'_{i_0}(t_{i_0}) - \delta^2 H_{i_0}(t_{i_0}) - (c_{i_0}(t_{i_0}) + \xi) H_{i_0}(t_{i_0}) - b_{i_0}(t_{i_0}) L_{i_0}(t_{i_0}) < 0,$$

but this inequality contradicts (2.1) and the proof is completed. □

Lemma 2.2. *Let* $(\underline{U_h}, \underline{V_h}, \underline{W_h}), (\overline{U_h}, \overline{V_h}, \overline{W_h}) \in \left(C^1([0,T_h), \mathbb{R}^I)\right)^3$ *be lower and upper solutions of* (1.2)–(1.5) *respectively such that,*
$(\underline{U_h}(0), \underline{V_h}(0), \underline{W_h}(0)) \leqslant (\overline{U_h}(0), \overline{V_h}(0), \overline{W_h}(0))$. *Then*

$$(\underline{U_h}(t), \underline{V_h}(t), \underline{W_h}(t)) \leqslant (\overline{U_h}(t), \overline{V_h}(t), \overline{W_h}(t)).$$

Proof. Let us define $(H_h(t), L_h(t), J_h(t)) = (\overline{U_h}(t), \overline{V_h}(t), \overline{W_h}(t)) - (\underline{U_h}(t), \underline{V_h}(t), \underline{W_h}(t))$.
By a straightforward computation and using the Mean value theorem, we obtain

$$H'_i(t) - \delta^2 H_i(t) - p_{11} a_i(\theta'_i(t))^{p_{11}-1} H_i(t) - p_{12} a_i(\mu'_i(t))^{p_{12}-1} L_i(t) \geqslant 0,\ i = 1, \ldots, I, \quad (2.5)$$

$$L'_i(t) - \delta^2 L_i(t) - p_{22} a_i(\mu'_i(t))^{p_{22}-1} L_i(t) - p_{23} a_i(\rho'_i(t))^{p_{23}-1} J_i(t) \geqslant 0,\ i = 1, \ldots, I, \quad (2.6)$$

$$J'_i(t) - \delta^2 J_i(t) - p_{33} a_i(\rho'_i(t))^{p_{33}-1} J_i(t) - p_{31} a_i(\theta'_i(t))^{p_{31}-1} H_i(t) \geqslant 0,\ i = 1, \ldots, I, \quad (2.7)$$

$$H_i(0) \geqslant 0,\ L_i(0) \geqslant 0,\ J_i(0) \geqslant 0, \quad i = 1, \ldots, I, \quad (2.8)$$

where $\theta'_i(t)$, $\mu'_i(t)$ and $\rho'_i(t)$ lie, respectively, between $\overline{U_i}(t)$ and $\underline{U_i}(t)$, between $\overline{V_i}(t)$ and $\underline{V_i}(t)$ and between $\overline{W_i}(t)$ and $\underline{W_i}(t)$ for $i = 1, \ldots, I$.
We can rewrite (2.5)–(2.7) as

$$\begin{aligned}
H'_i(t) - \delta^2 H_i(t) - c_i(t) H_i(t) - b_i(t) L_i(t) \geqslant 0, &\quad i = 1, \ldots, I,\ t \in (0, T_h),\\
L'_i(t) - \delta^2 L_i(t) - \theta_i(t) L_i(t) - \kappa_i(t) J_i(t) \geqslant 0, &\quad i = 1, \ldots, I,\ t \in (0, T_h),\\
J'_i(t) - \delta^2 J_i(t) - \mu_i(t) K_i(t) - \rho_i(t) H_i(t) \geqslant 0, &\quad i = 1, \ldots, I,\ t \in (0, T_h),
\end{aligned}$$

where $c_i(t) = p_{11} d_i(\theta'_i(t))^{p_{11}-1}$, $b_i(t) = p_{12} a_i(\mu'_i(t))^{p_{12}-1}$, $\theta_i(t) = p_{22} a_i(\mu'_i(t))^{p_{22}-1}$, $\kappa_i(t) = p_{23} a_i(\rho'_i(t))^{p_{23}-1}$, $\mu_i(t) = p_{33} a_i(\rho'_i(t))^{p_{33}-1}$ and $\rho_i(t) = p_{31} a_i(\theta'_i(t))^{p_{31}-1}$, for $i = 1, \ldots, I$, $\forall t \in (0, T_h)$. According to Lemma 2.1, $H_i(t) \geqslant 0$, $L_i(t) \geqslant 0$, $J_i(t) \geqslant 0$, for $i = 1, \ldots, I$, $\forall t \in (0, T_h)$ and the proof is completed. □

The next lemma gives the properties of the semidiscrete solution.

Lemma 2.3. *Let* $(U_h, V_h, W_h) \in \left(C^1([0,T_h), \mathbb{R}^I)\right)^3$ *be the solution of* (1.2)–(1.5) *with an initial data*
$(\psi_{1,h}, \psi_{2,h}, \psi_{3,h})$ *lower solution such that* $0 < \psi_{1,i} < \psi_{1,i+1}$, $0 < \psi_{2,i} < \psi_{2,i+1}$ *and* $0 < \psi_{3,i} < \psi_{3,i+1}$ *for* $i = 1, \ldots, I-1$. *Then we have*

(i): $(U_i(t),V_i(t),W_i(t)) \geq (\psi_{1,i},\psi_{2,i},\psi_{3,i}) > 0$, $i=1,\dots,I$, $t\in[0,T_h)$;
(ii): $(U_{i+1}(t),V_{i+1}(t),W_{i+1}(t)) > (U_i(t),V_i(t),W_i(t))$, $i=1,\dots,I-1$, $t\in[0,T_h)$;
(iii): $(U_i'(t),V_i'(t),W_i'(t)) > 0$, $i=1,\dots,I$, $t\in[0,T_h)$.

Proof. (i) Since $(\psi_{1,h},\psi_{2,h},\psi_{3,h})$ is a lower solution of (1.2)–(1.5), by the Lemma 2.2 we have
$(U_i(t),V_i(t),W_i(t)) \geq (\psi_{1,i},\psi_{2,i},\psi_{3,i}) > 0$, $i=1,\dots,I$, $t\in[0,T_h)$.

(ii) We argue by contradiction. Assume that t_0 is the first $t>0$, such that $(G_i,Q_i,M_i)(t) = (U_{i+1}-U_i, V_{i+1}-V_i, W_{i+1}-W_i)(t) > 0$, for $1\leqslant i\leqslant I-1$, but $G_{i_0}(t_0) = U_{i_0+1}(t_0)-U_{i_0}(t_0) = 0$ or $Q_{i_0}(t_0) = V_{i_0+1}(t_0)-V_{i_0}(t_0)=0$ or $M_{i_0}(t_0) = W_{i_0+1}(t_0) - W_{i_0}t_0) = 0$ for a certain $i_0\in\{1,\dots,I-1\}$, $t\in(0,t_0)$. Assume that $G_{i_0}(t_0) = U_{i_0+1}(t_0)-U_{i_0}(t_0)=0$. Without loss of generality, we assume that i_0 is the smallest integer which satisfies the above equality. We have

$$G_1'(t) = \frac{U_1(t)-2U_2(t)+U_3(t)}{h^2} - \left(\frac{2U_2(t)-2U_1(t)}{h^2} - \frac{2}{h}\left(U_1^{p_{11}}(t)+V_1^{p_{12}}(t)\right)\right),$$

$$G_i'(t) = \frac{U_i(t)-2U_{i+1}(t)+U_{i+2}(t)}{h^2} - \frac{U_{i-1}(t)-2U_i(t)+U_{i+1}(t)}{h^2},\ 2\leqslant i\leqslant I-2,$$

$$G_{I-1}'(t) = \frac{2U_{I-1}(t)-2U_I(t)}{h^2} - \frac{U_{I-2}(t)-2U_{I-1}(t)+U_I(t)}{h^2} + \frac{2}{h}\left(U_I^{p_{11}}(t)+V_I^{p_{12}}(t)\right),$$

$$\begin{cases} G_1'(t) = \dfrac{G_2(t)-3G_1(t)}{h^2} + \dfrac{2}{h}\left(U_1^{p_{11}}(t)+V_1^{p_{12}}(t)\right), \\ G_i'(t) = \dfrac{G_{i-1}(t)-2G_i(t)+G_{i+1}(t)}{h^2},\ 2\leqslant i\leqslant I-2, \\ G_{I-1}'(t) = \dfrac{G_{I-2}(t)-3G_{I-1}(t)}{h^2} + \dfrac{2}{h}\left(U_I^{p_{11}}(t)+V_I^{p_{12}}(t)\right). \end{cases} \tag{2.9}$$

According to the hypotheses on t_0, we have the following inequalities

$$G_{i_0}'(t_0) = \lim_{\epsilon\to 0}\frac{G_{i_0}(t_0)-G_{i_0}(t_0-\epsilon)}{\epsilon} \leqslant 0,$$

$$\frac{G_{i_0-1}(t_0)-2G_{i_0}(t_0)+G_{i_0+1}(t_0)}{h^2} > 0, \text{ if } \quad 2\leqslant i_0\leqslant I-2,$$

$$\frac{G_{i_0+1}(t_0)-3G_{i_0}(t_0)}{h^2} > 0 \text{ if } i_0 = 1,$$

$$\frac{-3G_{i_0}(t_0)+G_{i_0-1}(t_0)}{h^2} > 0 \text{ if } i_0 = I-1,$$

which implies,

$$G_{i_0}'(t_0) - \frac{G_{i_0-1}(t_0)-2G_{i_0}(t_0)+G_{i_0+1}(t_0)}{h^2} < 0, \text{ if } \quad 2\leqslant i_0\leqslant I-2,$$

$$G_{i_0}'(t_0) - \frac{G_{i_0+1}(t_0)-3G_{i_0}(t_0)}{h^2} - \frac{2}{h}\left(U_{i_0}^{p_{11}}(t)+V_{i_0}^{p_{12}}(t)\right) < 0, \quad \text{if} \quad i_0 = 1,$$

$$G'_{i_0}(t_0) - \frac{-3G_{i_0}(t_0) + G_{i_0-1}(t_0)}{h^2} - \frac{2}{h}\left(U_{i_0}^{p_{11}}(t) + V_{i_0}^{p_{12}}(t)\right) < 0, \quad \text{if} \quad i_0 = I-1.$$

Therefore, we have a contradiction with (2.9), which leads to the desired result.

(iii) Denote $R_i(t) = U'_i(t)$, $S_i(t) = V'_i(t)$ and $O_i(t) = W'_i(t)$, for $i = 1, \dots, I$ and $t \in [0, T_h)$. Let t_0 be the first $t \in [0, T_h)$ such as $R_i(t) > 0$, $S_i(t) > 0$ and $O_i(t) > 0$ for $t \in (0, t_0)$, but $R_{i_0}(t_0) = 0$, $S_{i_0}(t_0) = 0$ and $O_{i_0}(t_0) = 0$ for a certain $i_0 \in \{1, \dots, I\}$. Without loss of generality, we suppose that i_0 is the smallest integer verifying the above inequality. We have

$$R'_{i_0}(t_0) = \lim_{\epsilon \to 0} \frac{R_{i_0}(t_0) - R_{i_0}(t_0 - \epsilon)}{\epsilon} \leqslant 0, \quad 1 \leqslant i_0 \leqslant I,$$

$$\delta^2 R_{i_0}(t_0) = \frac{R_{i_0-1}(t_0) - 2R_{i_0}(t_0) + R_{i_0+1}(t_0)}{h^2} > 0, \text{ if } \quad 2 \leqslant i_0 \leqslant I-2,$$

$$\delta^2 R_{i_0}(t_0) = \frac{2R_2(t_0) - 2R_1(t_0)}{h^2} > 0 \text{ if } i_0 = 1,$$

$$\delta^2 R_{i_0}(t_0) = \frac{2R_{I-2}(t_0) - 2R_{I-1}(t_0)}{h^2} > 0 \text{ if } i_0 = I-1,$$

which implies,

$$R'_{i_0}(t_0) - \delta^2 R_{i_0}(t_0) - d_{i_0}\left(U_{i_0}^{p_{11}}(t) + V_{i_0}^{p_{12}}(t)\right) < 0, \text{ if } \quad 1 \leqslant i_0 \leqslant I-1.$$

This contradicts (1.2)–(1.5). This ends the proof. □

3. Semidiscrete blow-up solution

In this section under some assumptions, we show that the solution (U_h, V_h, W_h) of (1.2)–(1.5) blows up in a finite time.

Definition 3.1. We say that the solution (U_h, V_h, W_h) of (1.2)–(1.5) blows up in a finite time if there exists a finite time $T_h > 0$ such that for all $t \in [0, T_h)$, $\max\{\|U_h(t)\|_\infty, \|V_h(t)\|_\infty, \|W_h(t)\|_\infty\} < +\infty$ and $\lim_{t \to T_h} (\|U_h(t)\|_\infty + \|V_h(t)\|_\infty + \|W_h(t)\|_\infty) = +\infty$, where $\|U_h(t)\|_\infty = \max_{1 \le i \le I} |U_i(t)|$, $\|V_h(t)\|_\infty = \max_{1 \le i \le I} |V_i(t)|$, $\|W_h(t)\|_\infty = \max_{1 \le i \le I} |W_i(t)|$. T_h is called the blow-up time of the solution (U_h, V_h, W_h).

In the continuation, we have the conditions of global existence of the solution of the semi-discrete problem.

Theorem 3.1. *The solution (U_h, V_h, W_h) of the system (1.2)–(1.5) exists globally if*

$$\max\{p_{11},\ p_{22},\ p_{33},\ p_{12}p_{23}p_{31}\} \le 1.$$

Proof. Suppose $\max\{p_{11},\ p_{22},\ p_{33},\ p_{12}p_{23}p_{31}\} \le 1$ and let $\overline{U}_h$, $\overline{V}_h$ and $\overline{W}_h$ such that

$$\overline{U}_i(t) = D\mathrm{e}^{k_1 t + (i-1)l_1 h}, \quad \overline{V}_i(t) = E\mathrm{e}^{k_2 t + (i-1)l_2 h}, \quad \overline{W}_i(t) = F\mathrm{e}^{k_3 t + (i-1)l_3 h},$$
$$i = 1, \dots, I,\ t \in (0, T_h),$$

where D, E, F, k_j, l_j $(j=1,2,3)$ are positive constants satisfying

$$D \geq \max_{1\leqslant i\leqslant I} |U_i(0)|,\ E \geq \max_{1\leqslant i\leqslant I} |V_i(0)|,\ F \geq \max_{1\leqslant i\leqslant I} |W_i(0)|,$$

$$\begin{cases} k_1 = k_2 p_{12} = k_3 p_{23} p_{12} \geq \max\{\Phi, M, Z\}, \\ l_1 = l_2 p_{12} = l_3 p_{23} p_{12}, \end{cases}$$

and

$$\begin{cases} k_2 = k_3 p_{23}, \\ l_2 = l_3 p_{23}, \end{cases}$$

where
$\Phi = \dfrac{2}{h^2}\left(\mathrm{e}^{l_1 h} - 1 + h\left(D^{p_{11}} + E^{p_{12}}\right)/D\right)$, $M = \dfrac{2}{h^2}\left(\mathrm{e}^{l_2 h} - 1 + h\left(E^{p_{22}} + F^{p_{23}}\right)/E\right)$,
$Z = \dfrac{2}{h^2}\left(\mathrm{e}^{l_3 h} - 1 + h\left(F^{p_{33}} + D^{p_{31}}\right)/F\right)$.
It is not hard to see that for $i = 1, \ldots, I$,

$$\begin{array}{rcl} \mathrm{e}^{k_1 t + (i-1) l_1 h} & = & \mathrm{e}^{k_2 p_{12} t + (i-1) l_2 p_{12} h} = \mathrm{e}^{k_3 p_{23} p_{12} t + (i-1) l_3 p_{23} p_{12} h}. \\ \mathrm{e}^{k_2 t + (i-1) l_2 h} & = & \mathrm{e}^{k_3 p_{23} t + (i-1) l_3 p_{23} h}. \\ \mathrm{e}^{k_3 t + (i-1) l_3 h} & \geq & \mathrm{e}^{k_3 p_{23} p_{12} p_{31} t + (i-1) l_3 p_{23} p_{12} p_{31} h} = \mathrm{e}^{k_1 p_{31} t + (i-1) l_1 p_{31} h}, \\ \textit{because} \quad (p_{23} p_{12} p_{31} & \leq & 1). \end{array}$$

$$D^{p_{11}} + E^{p_{12}} \geq D^{p_{11}} \mathrm{e}^{k_1 (p_{11}-1) t + (i-1) l_1 (p_{11}-1) h} + E^{p_{12}} \quad \textit{because} \quad (p_{11} \leq 1),$$

$$2\frac{\left(D^{p_{11}} + E^{p_{12}}\right)/D}{h} \overline{U}_i(t) \geq 2\frac{\left(D^{p_{11}} \mathrm{e}^{k_1 (p_{11}-1) t + (i-1) l_1 (p_{11}-1) h} + E^{p_{12}}\right)/D}{h} \overline{U}_i(t),$$

$$2\frac{\left(D^{p_{11}} + E^{p_{12}}\right)/D}{h} \overline{U}_i(t) \geq$$

$$\frac{\left(2D^{p_{11}} \mathrm{e}^{k_1 p_{11} t + (i-1) l_1 p_{11} h} + 2E^{p_{12}} \mathrm{e}^{k_1 t + (i-1) l_1 h}\right)/D}{h} \mathrm{e}^{-k_1 t - (i-1) l_1 h} \overline{U}_i(t),$$

$$2\frac{\left(D^{p_{11}} + E^{p_{12}}\right)/D}{h} \overline{U}_i(t) \geq \frac{2D^{p_{11}} \mathrm{e}^{k_1 p_{11} t + (i-1) l_1 p_{11} h} + 2E^{p_{12}} \mathrm{e}^{k_1 t + (i-1) l_1 h}}{h},$$

$$2\frac{\left(D^{p_{11}} + E^{p_{12}}\right)/D}{h} \overline{U}_i(t) \geq \frac{2D^{p_{11}} \mathrm{e}^{k_1 p_{11} t + (i-1) l_1 p_{11} h} + 2E^{p_{12}} \mathrm{e}^{k_2 p_{12} t + (i-1) l_2 p_{12} h}}{h},$$

$$2\frac{\left(D^{p_{11}} + E^{p_{12}}\right)/D}{h} \overline{U}_i(t) \geq \frac{2\overline{U}_i^{p11}(t) + 2\overline{V}_i^{p12}(t)}{h}.$$

By the same way, we prove that for $i = 1, \ldots, I$,

$$2\frac{\left(E^{p_{22}} + F^{p_{23}}\right)/E}{h} \overline{V}_i(t) \geq \frac{2\overline{V}_i^{p22}(t) + 2\overline{W}_i^{p23}(t)}{h}$$

and

$$2\frac{\left(F^{p_{33}} + D^{p_{31}}\right)/F}{h} \overline{W}_i(t) \geq \frac{2\overline{W}_i^{p33}(t) + 2\overline{U}_i^{p31}(t)}{h}.$$

We have

$$\begin{aligned}
\delta^2\overline{U}_1(t) &= \frac{2\overline{U}_2(t) - 2\overline{U}_1(t)}{h^2},\ t \in (0, T_h),\\
\delta^2\overline{U}_i(t) &= \frac{\overline{U}_{i-1}(t) - 2\overline{U_i(t)} + \overline{U}_{i+1}(t)}{h^2},\ 2 \leqslant i \leqslant I-1,\ t \in (0, T_h),\\
\delta^2\overline{U}_I(t) &= \frac{2\overline{U}_{I-1}(t) - 2\overline{U}_I(t)}{h^2},\ t \in (0, T_h),
\end{aligned}$$

$$\delta^2\overline{U}_1(t) = \frac{2D\mathrm{e}^{k_1t+(1+1-1)l_1h} - 2D\mathrm{e}^{k_1t+(1-1)l_1h}}{h^2},\ t \in (0, T_h),$$

$$\delta^2\overline{U}_i(t) = \frac{D\mathrm{e}^{k_1t+(i-1-1)l_1h} - 2D\mathrm{e}^{k_1t+(i-1)l_1h} + D\mathrm{e}^{k_1t+(i+1-1)l_1h}}{h^2},$$
$$2 \leqslant i \leqslant I-1,\ t \in (0, T_h),$$

$$\delta^2\overline{U}_I(t) = \frac{2D\mathrm{e}^{k_1t+(I-1-1)l_1h} - 2D\mathrm{e}^{k_1t+(I-1)l_1h}}{h^2},\ t \in (0, T_h),$$

$$\begin{aligned}
\delta^2\overline{U}_1(t) &= 2D\mathrm{e}^{k_1t+(1-1)l_1h}\frac{\mathrm{e}^{l_1h} - 1}{h^2},\ t \in (0, T_h),\\
\delta^2\overline{U}_i(t) &= 2D\mathrm{e}^{k_1t+(i-1)l_1h}\frac{\mathrm{e}^{-l_1h} - 2 + \mathrm{e}^{l_1h}}{h^2},\ 2 \leqslant i \leqslant I-1,\ t \in (0, T_h),\\
\delta^2\overline{U}_I(t) &= 2D\mathrm{e}^{k_1t+(I-1)l_1h}\frac{\mathrm{e}^{-l_1h} - 1}{h^2},\ t \in (0, T_h),
\end{aligned}$$

$$\begin{aligned}
\delta^2\overline{U}_1(t) &= 2\frac{\mathrm{e}^{l_1h} - 1}{h^2}\overline{U}_1(t),\ t \in (0, T_h),\\
\delta^2\overline{U}_i(t) &= 2\frac{\cosh(l_1h) - 1}{h^2}\overline{U}_i(t),\ 2 \leqslant i \leqslant I-1,\ t \in (0, T_h),\\
\delta^2\overline{U}_I(t) &= 2\frac{\mathrm{e}^{-l_1h} - 1}{h^2}\overline{U}_I(t),\ t \in (0, T_h).
\end{aligned}$$

By the same way, we also prove that

$$\begin{aligned}
\delta^2\overline{V}_1(t) &= 2\frac{\mathrm{e}^{l_2h} - 1}{h^2}\overline{V}_1(t),\ t \in (0, T_h),\\
\delta^2\overline{V}_i(t) &= 2\frac{\cosh(l_2h) - 1}{h^2}\overline{V}_i(t),\ 2 \leqslant i \leqslant I-1,\ t \in (0, T_h),\\
\delta^2\overline{V}_I(t) &= 2\frac{\mathrm{e}^{-l_2h} - 1}{h^2}\overline{V}_I(t),\ t \in (0, T_h).
\end{aligned}$$

And

$$\begin{aligned}
\delta^2\overline{W}_1(t) &= 2\frac{\mathrm{e}^{l_3h} - 1}{h^2}\overline{W}_1(t),\ t \in (0, T_h),\\
\delta^2\overline{W}_i(t) &= 2\frac{\cosh(l_3h) - 1}{h^2}\overline{W}_i(t),\ 2 \leqslant i \leqslant I-1,\ t \in (0, T_h),\\
\delta^2\overline{W}_I(t) &= 2\frac{\mathrm{e}^{-l_3h} - 1}{h^2}\overline{W}_I(t),\ t \in (0, T_h).
\end{aligned}$$

For $i = 1, \dots, I$,

$$\overline{U_i}'(t) = k_1 D \mathrm{e}^{k_1 t + (i-1) l_1 h}, \quad \overline{V_i}'(t) = k_2 E \mathrm{e}^{k_2 t + (i-1) l_2 h}, \quad \overline{W_i}'(t) = k_3 F \mathrm{e}^{k_3 t + (i-1) l_3 h}, \quad t \in (0, T_h),$$

which implies that for $i = 1, \dots, I$,

$$\overline{U_i}'(t) = k_1 \overline{U}_i(t), \quad \overline{V_i}'(t) = k_2 \overline{V}_i(t), \quad \overline{W_i}'(t) = k_3 \overline{W}_i(t), t \in (0, T_h),$$

then for $i = 1, \dots, I$,

$$\overline{U_i}'(t) \geq \Phi \overline{U}_i(t), \quad \overline{V_i}'(t) \geq M \overline{V}_i(t), \quad \overline{W_i}'(t) \geq Z \overline{W}_i(t), t \in (0, T_h),$$

which implies that for $i = 1, \dots, I$,

$$\begin{aligned}
\overline{U_i}'(t) &\geq \frac{2}{h^2} \left(\mathrm{e}^{l_1 h} - 1 + h \left(D^{p_{11}} + E^{p_{12}} \right) / D \right) \overline{U}_i(t), \qquad t \in (0, T_h), \\
\overline{V_i}'(t) &\geq \frac{2}{h^2} \left(\mathrm{e}^{l_2 h} - 1 + h \left(E^{p_{22}} + F^{p_{23}} \right) / E \right) \overline{V}_i(t), \qquad t \in (0, T_h), \\
\overline{W_i}'(t) &\geq \frac{2}{h^2} \left(\mathrm{e}^{l_3 h} - 1 + h \left(F^{p_{33}} + D^{p_{31}} \right) / F \right) \overline{W}_i(t), \qquad t \in (0, T_h),
\end{aligned}$$

which implies that for $i = 1, \dots, I$,

$$\begin{aligned}
\overline{U_i}'(t) &\geq \frac{2}{h^2} \left(\mathrm{e}^{l_1 h} - 1 \right) \overline{U}_i(t) + \frac{2}{h} \left(\left(D^{p_{11}} + E^{p_{12}} \right) / D \right) \overline{U}_i(t), \qquad t \in (0, T_h), \\
\overline{V_i}'(t) &\geq \frac{2}{h^2} \left(\mathrm{e}^{l_2 h} - 1 \right) \overline{V}_i(t) + \frac{2}{h} \left(\left(E^{p_{22}} + F^{p_{23}} \right) / E \right) \overline{V}_i(t), \qquad t \in (0, T_h), \\
\overline{W_i}'(t) &\geq \frac{2}{h^2} \left(\mathrm{e}^{l_3 h} - 1 \right) \overline{W}_i(t) + \frac{2}{h} \left(\left(F^{p_{33}} + D^{p_{31}} \right) / F \right) \overline{W}_i(t), \qquad t \in (0, T_h).
\end{aligned}$$

As

$$\begin{cases}
\mathrm{e}^{l_1 h} - 1 \geq \cosh(l_1 h) - 1 \geq \mathrm{e}^{-l_1 h} - 1, \\
\mathrm{e}^{l_2 h} - 1 \geq \cosh(l_2 h) - 1 \geq \mathrm{e}^{-l_2 h} - 1, \\
\mathrm{e}^{l_3 h} - 1 \geq \cosh(l_3 h) - 1 \geq \mathrm{e}^{-l_3 h} - 1,
\end{cases}$$

and from (**??**), (**??**) and (**??**), we obtain

$$\begin{aligned}
\overline{U_i}'(t) &\geq \delta^2 \overline{U}_i(t) + \frac{2}{h} \left(\overline{U}_i^{p11}(t) + \overline{V}_i^{p12}(t) \right), \qquad i = 1, \dots, I, \ t \in (0, T_h), \\
\overline{V_i}'(t) &\geq \delta^2 \overline{V}_i(t) + \frac{2}{h} \left(\overline{V}_i^{p22}(t) + \overline{W}_i^{p23}(t) \right), \qquad i = 1, \dots, I, \ t \in (0, T_h), \\
\overline{W_i}'(t) &\geq \delta^2 \overline{W}_i(t) + \frac{2}{h} \left(\overline{W}_i^{p33}(t) + \overline{U}_i^{p31}(t) \right), \qquad i = 1, \dots, I, \ t \in (0, T_h).
\end{aligned}$$

Therefore, we have the following inequalities

$$\begin{aligned}
\overline{U_i}'(t) &\geq \delta^2 \overline{U}_i(t) + a_i \left(\overline{U}_i^{p11}(t) + \overline{V}_i^{p12}(t) \right), \qquad i = 1, \dots, I, \ t \in (0, T_h), \\
\overline{V_i}'(t) &\geq \delta^2 \overline{V}_i(t) + a_i \left(\overline{V}_i^{p22}(t) + \overline{W}_i^{p23}(t) \right), \qquad i = 1, \dots, I, \ t \in (0, T_h), \\
\overline{W_i}'(t) &\geq \delta^2 \overline{W}_i(t) + a_i \left(\overline{W}_i^{p33}(t) + \overline{U}_i^{p31}(t) \right), \qquad i = 1, \dots, I, \ t \in (0, T_h), \\
\overline{U}_i(0) &\geq U_i(0), \quad \overline{V}_i(0) \geq V_i(0), \quad \overline{W}_i(0) \geq W_i(0), \qquad i = 1, \dots, I.
\end{aligned}$$

From Lemma 2.2, we conclude that $(\overline{U}_h, \overline{V}_h, \overline{W}_h)$ is a upper solution of (1.2)–(1.5), hence the global existence of the solution of (1.2)–(1.5) and the proof is complete. □

Theorem 3.2. *If* $\max\{p_{11},\ p_{22},\ p_{33},\ p_{12}p_{23}p_{31}\} > 1$*, then the solution* (U_h, V_h, W_h) *of* (1.2)–(1.5) *blows up in a finite time* T_h*.*

Proof. Let (U_h, V_h, W_h) be the solution of the semidiscrete problem (1.2)–(1.5). If $p_{11} > 1$, $p_{22} > 1$ or $p_{33} > 1$ and if there exists a real $\alpha > 0$ such that

$$\delta^2\psi_{1,i} + a_i\psi_{1,i}^{p_{11}} + a_i\psi_{2,i}^{p_{12}} \geq \alpha(a_i\psi_{1,i}^{p_{11}} + a_i\psi_{2,i}^{p_{12}}), \quad i = 1, \ldots, I, \tag{3.1}$$
$$\delta^2\psi_{2,i} + a_i\psi_{2,i}^{p_{22}} + a_i\psi_{3,i}^{p_{23}} \geq \alpha(a_i\psi_{2,i}^{p_{22}} + a_i\psi_{3,i}^{p_{23}}), \quad i = 1, \ldots, I, \tag{3.2}$$
$$\delta^2\psi_{3,i} + a_i\psi_{3,i}^{p_{33}} + a_i\psi_{1,i}^{p_{31}} \geq \alpha(a_i\psi_{3,i}^{p_{33}} + a_i\psi_{1,i}^{p_{31}}), \quad i = 1, \ldots, I, \tag{3.3}$$

then (U_h, V_h, W_h) blows up in a finite time T_h.
We only demonstrate case where $p_{11} > 1$.
Let the functions $l_h(t)$, $m_h(t)$ and $n_h(t)$ be defined by

$l_h(t) = U_h'(t) - \alpha((U_h(t))^{p_{11}} + (V_h(t))^{p_{12}}),\ t \in (0, T_h),$
$m_h(t) = V_h'(t) - \alpha((V_h(t))^{p_{22}} + (W_h(t))^{p_{23}}),\ t \in (0, T_h),$
$n_h(t) = W_h'(t) - \alpha((W_h(t))^{p_{33}} + (U_h(t))^{p_{31}}),\ t \in (0, T_h).$

With a straightforward calculation we get

$l_i'(t) - \delta^2 l_i(t) - p_{11}a_i(U_i(t))^{p_{11}-1}l_i(t) - p_{12}a_i((V_i(t))^{p_{12}-1}m_i(t) \geq 0, i = 1, \ldots, I,\ t \in (0, T_h),$
$m_i'(t) - \delta^2 m_i(t) - p_{22}a_i(V_i(t))^{p_{22}-1}m_i(t) - p_{23}a_i((W_i(t))^{p_{23}-1}n_i(t) \geq 0, i = 1, \ldots, I,\ t \in (0, T_h),$
$n_i'(t) - \delta^2 n_i(t) - p_{33}a_i(W_i(t))^{p_{33}-1}n_i(t) - p_{31}a_i((U_i(t))^{p_{31}-1}l_i(t) \geq 0, i = 1, \ldots, I,\ t \in (0, T_h).$

Lemma 2.1 permits us to obtain $l_h(t) \geq 0$, $m_h(t) \geq 0$ and $n_h(t) \geq 0$.
Assume $\max\{p_{11},\ p_{22},\ p_{33},\ p_{12}p_{23}p_{31}\} = p_{11} > 1$, since $l_h(t) \geq 0$, we obtain the following inequality

$$\frac{d\|U_h(t)\|_\infty}{dt} \geq \alpha\|U_h(t)\|_\infty^{p_{11}}.$$

Hence U_h blows up in a finite time T_h and integrating (21) from 0 to T_h we prove that $T_h \leq \dfrac{\|\psi_{1,h}\|_\infty^{1-p_{11}}}{\alpha(p_{11}-1)}$.
Analogously, we show that V_h and W_h blow up in a finite time T_h if $p_{22} > 1$ and $p_{33} > 1$. Then, we get $T_h \leq \dfrac{\|\psi_{2,h}\|_\infty^{1-p_{22}}}{\alpha(p_{22}-1)}$ and $T_h \leq \dfrac{\|\psi_{3,h}\|_\infty^{1-p_{33}}}{\alpha(p_{33}-1)}$.
Therefore if $p_{11} > 1$, $p_{22} > 1$, $p_{33} > 1$, then (U_h, V_h, W_h) blows up in a finite time T_h and we have

$$T_h \leq \min\left\{\frac{\|\psi_{1,h}\|_\infty^{1-p_{11}}}{\alpha(p_{11}-1)}, \frac{\|\psi_{2,h}\|_\infty^{1-p_{22}}}{\alpha(p_{22}-1)}, \frac{\|\psi_{3,h}\|_\infty^{1-p_{33}}}{\alpha(p_{33}-1)}\right\}.$$

Now, assume that $\max\{p_{11},\ p_{22},\ p_{33},\ p_{12}p_{23}p_{31}\} = p_{12}p_{23}p_{31} > 1$.
We may assume without loss of generality that $\inf_{1\le i\le I}\psi_{j,i} \ge c > 0$ $(j = 1, 2, 3)$. Let $(\beta_1, \beta_2, \beta_3)^T$ be the solution of

$$\begin{pmatrix} 1 & -p_{12} & 0 \\ 0 & 1 & -p_{23} \\ -p_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and denote $T_0 = S\Theta^{-1} - 1$, $\underline{U_i}(t) = (S - \Theta - \Theta t)^{-\beta_1}$, $\underline{V_i}(t) = (S - \Theta - \Theta t)^{-\beta_2}$, $\underline{W_i}(t) = (S - \Theta - \Theta t)^{-\beta_3}$ for $i = 1, \dots, I$, $t \in [0, T_0]$ (S, Θ are positive constants to be determined). A simple computation shows for $i = 1, \dots, I,\ t \in [0, T_0]$. Denote $-\beta_2 p_{12} = -\beta_1 - 1$, $-\beta_3 p_{23} = -\beta_2 - 1$ and $-\beta_1 p_{31} = -\beta_3 - 1$.

$$\underline{U_i}'(t) - \delta^2 \underline{U_i}(t) - d_i(\underline{U_i}^{p_{11}}(t) + \underline{V_i}^{p_{12}}(t)) =$$
$$(\beta_1\Theta - \frac{2}{h})(S - \Theta - \Theta t)^{-\beta_2 p_{12}} - \frac{2}{h}(S - \Theta - \Theta t)^{-\beta_1 p_{11}},$$

$$\underline{V_i}'(t) - \delta^2 \underline{V_i}(t) - d_i(\underline{V_i}^{p_{22}}(t) + \underline{W_i}^{p_{23}}(t)) =$$
$$(\beta_2\Theta - \frac{2}{h})(S - \Theta - \Theta t)^{-\beta_3 p_{23}} - \frac{2}{h}(S - \Theta - \Theta t)^{-\beta_2 p_{22}},$$

$$\underline{W_i}'(t) - \delta^2 \underline{W_i}(t) - d_i(\underline{W_i}_i^{p_{33}}(t) + \underline{U_i}^{p_{31}}(t)) =$$
$$(\beta_3\Theta - \frac{2}{h})(S - \Theta - \Theta t)^{-\beta_3 p_{31}} - \frac{2}{h}(S - \Theta - \Theta t)^{-\beta_3 p_{33}},$$

$$\underline{U_i}(0) = (S - \Theta)^{-\beta_1},\ \underline{V_i}(0) = (S - \Theta)^{-\beta_2},\ \underline{W_i}(0) = (S - \Theta)^{-\beta_3}.$$

Take Θ to be so small $\beta_j\Theta \le \dfrac{2}{h}$ and S to be so large such that $S \ge \Theta + c^{-\frac{1}{\beta_j}}$, $(j = 1, 2, 3)$, we have

$$\begin{aligned} \underline{U_i}'(t) - \delta^2 \underline{U_i}(t) - d_i(\underline{U_i}^{p_{11}}(t) + \underline{V_i}^{p_{12}}(t)) &\le 0, \\ \underline{V_i}'(t) - \delta^2 \underline{V_i}(t) - d_i(\underline{V_i}^{p_{22}}(t) + \underline{W_i}^{p_{23}}(t)) &\le 0, \\ \underline{W_i}'(t) - \delta^2 \underline{W_i}(t) - d_i(\underline{W_i}_i^{p_{33}}(t) + \underline{U_i}^{p_{31}}(t)) &\le 0, \end{aligned}$$

$$0 < \underline{U_i}(0) \le (S - \Theta)^{-\beta_1},\ 0 < \underline{V_i}(0) \le (S - \Theta)^{-\beta_2},\ 0 < \underline{W_i}(0) \le (S - \Theta)^{-\beta_3}.$$

then $\left(\underline{U_h}, \underline{V_h}, \underline{W_h}\right)$ is a lower solution of (1.2)–(1.5). Since $\max\limits_{t\to T_0}(\underline{U_i}(t), \underline{V_i}(t), \underline{W_i}(t)) \to +\infty$. Therefore $\left(\underline{U_h}, \underline{V_h}, \underline{W_h}\right)$ blows up. This ends the proof. $\square$

Remark 3.1. By integrating the inequality (21) over (t, T_h), we have

$$\frac{1}{p_{11} - 1} \frac{1}{\|U_h(t)\|_\infty^{p_{11}-1}} \ge \alpha\,(T_h - t)$$

and there exists a constant $C_{11} > 0$ such that

$$\|U_h(t)\|_\infty \leq C_{11}\,(T_h - t)^{-\frac{1}{p_{11}-1}}\,,\ t \in (0, T_h)$$

where $p_{11} > 1$.
For $p_{22} > 1$ and $p_{33} > 1$, there exists constants $C_{22} > 0$ and $C_{33} > 0$ such that

$$\|V_h(t)\|_\infty \leq C_{22}\,(T_h - t)^{-\frac{1}{p_{22}-1}}\,,\ t \in (0, T_h)$$

where $p_{22} > 1$.

$$\|W_h(t)\|_\infty \leq C_{33}\left(T_h^b - t\right)^{-\frac{1}{p_{33}-1}}\,,\ t \in (0, T_h)$$

where $p_{33} > 1$.

4. Non-simultaneous blow-up

We consider (U_h, V_h, W_h) to be the solution of (1.2)–(1.5).

Definition 4.1. We say that the solution (U_h, V_h, W_h) of (1.2)–(1.5) blows up non-simultaneously in a finite time $T_h > 0$ when one or two of its three components blow up at T_h while the others remain bounded on $[0, T_h)$.

4.1. **Only one component blowing up.** In this subsection, we give the results on only one component blowing up for every initial data.

Theorem 4.1. *Assume $p_{11} \leq 1$, $p_{22} \leq 1$ and $p_{23} + 1 < p_{33}$. Then only W_h blows up for every positive initial data.*

Theorem 4.2. *Assume $p_{11} \leq 1$, $p_{33} \leq 1$ and $p_{12} + 1 < p_{22}$. Then only V_h blows up for every positive initial data.*

Theorem 4.3. *Assume $p_{22} \leq 1$, $p_{33} \leq 1$ and $p_{31} + 1 < p_{11}$. Then only U_h blows up for every positive initial data.*

We only prove the theorem 4.1 (the case for W_h blowing up and U_h,V_h remaining bounded). Theorems 4.2 and 4.3 prove themselves in the same way as theorem 4.1.

Proof. If W_h remains bounded, then V_h would remain bounded for $p_{22} \leq 1$, and hence U_h would remain bounded for $p_{11} \leq 1$. This is a contradiction to the blow-up property of solution (U_h, V_h, W_h) since $p_{33} > 1$.
Consequently W_h blows up in T_h and of the remark 3.1, we have for $i = 1, \ldots, I$

$$W_i(t) \leq C_{33}\,(T_h - t)^{-\dfrac{1}{p_{33}-1}}\,, \quad \forall t \in (0, T_h).$$

Suppose that V_h blows up and I blowing up node. Then, there exists $K, C > 0$, $t_0 \in (0, T_h)$ such that

$$V_I'(t) \leq K V_I^{p_{22}}(t) + C\,(T_h - t)^{-\dfrac{p_{23}}{p_{33}-1}}\,, \quad \forall t \in (t_0, T_h),$$

which implies that

$$\begin{aligned}
\frac{V_I'(t)}{V_I^{p_{22}}(t)} &\leq \frac{C\left(T_h-t\right)^{-\dfrac{p_{23}}{p_{33}-1}}}{V_I^{p_{22}}(t)}+K, \quad \forall t \in (t_0,T_h),\\
\frac{V_I'(t)}{V_I^{p_{22}}(t)} &\leq C\left(T_h-t\right)^{-\dfrac{p_{23}}{p_{33}-1}}+K, \quad \forall t \in (t_0,T_h) \quad \textit{because} \quad \left(V_I(t)\geq 1\right),
\end{aligned}$$

integrating this inequality from t_0 to t, we obtain

$$V_I(t) \leq \exp\left(\ln\left(V_I(t_0)\right)+C_1 T_h^{\dfrac{p_{33}-p_{23}-1}{p_{33}-1}}+KT_h\right), \quad \forall t \in (t_0,T_h), \textit{and} \quad p_{22}=1,$$

where $C_1=\dfrac{C\left(p_{33}-1\right)}{p_{33}-p_{23}-1}$ and

$$V_I(t) \leq \left(V_I^{1-p_{22}}(t_0)+C_2 T_h^{\dfrac{p_{33}-p_{23}-1}{p_{33}-1}}+K_1 T_h\right)^{\dfrac{1}{1-p_{22}}}, \quad \forall t \in (t_0,T_h), \textit{and} \quad p_{22}<1,$$

where $C_2=\left(1-p_{22}\right)C_1$ et $K_1=\left(1-p_{22}\right)K$.
Which is a contradiction with V_h blows up and the proof is completed. □

4.2. **Only one component remaining bounded.** In this subsection, we discuss the non-simultaneous blow-up for only one component remaining bounded up to the blow-up time T_h.

Theorem 4.4. *The case for only U_h remaining bounded up to T_h.*
If $1<p_{33}<p_{23}+1$, $p_{11}\leq 1$, $p_{22}\leq 1$, and $p_{12}<\dfrac{p_{33}-1}{p_{23}+1-p_{33}}$, then, for every initial data, V_h and W_h blow up simultaneously at T_h while U_h remains bounded up to time T_h.

Proof. This proof consists of three steps.
Step 1. V_h and W_h blow up simultaneously at T_h.
First, assume that W_h remains bounded up to time T_h, since $p_{11},\ p_{22}\leq 1$, U_h and V_h also remain bounded. This is a contradiction to the blow-up property of solution (U_h,V_h,W_h) for $p_{33}>1$.
Next, assume that V_h remains bounded up to time T_h. Since $p_{11}\leq 1$, there exists a constant $E>0$ such that $U_i(t)\leq E$ for $i=1,\ldots,I$, $\forall t\in(0,T_h)$, then W_h satisfies

$$W_I'(t)\leq \frac{2}{h}\left(W_I^{p_{33}}(t)+E^{p_{31}}\right), \quad \forall t\in(0,T_h),$$

because $\forall t\in(0,T_h)$, $W_i(t)\leq W_{i+1}(t)$ for $i=1,\ldots,I$ (Lemma 2.3).
This implies that

$$\frac{W_I'(t)}{W_I^{p_{33}}(t)}\leq \frac{2}{h}\left(1+\frac{E^{p_{31}}}{W_I^{p_{33}}(t)}\right), \quad \forall t\in(0,T_h),$$

which implies that

$$\frac{W_I'(t)}{W_I^{p_{33}}(t)} \leq \frac{2}{h}\left(1+E^{p_{31}}\right), \quad \forall t \in (0,T_h), \quad \textit{because} \quad \left(W_I(t) \geq 1\right).$$

Integrating this inequality from t to T_h, we obtain

$$W_I(t) \geq c\,(T_h-t)^{-\dfrac{1}{p_{33}-1}}, \quad \forall t \in (0,T_h),$$

where $c = \left(\dfrac{2}{h}\left(1+E^{p_{31}}\right)(p_{33}-1)\right)^{-\dfrac{1}{p_{33}-1}}$.
From (1.3) and (**??**), we have

$$V_I'(t) \geq \frac{2V_{I-1}(t)-2V_I(t)}{h^2} + \frac{2}{h}\left(V_I^{p_{22}}(t) + c^{p_{23}}\,(T_h-t)^{-\dfrac{p_{23}}{p_{33}-1}}\right),\ t \in (0,T_h),$$

as V_h is bounded, then there exists a constant K_1 dependent of h such that

$$\begin{aligned} V_I'(t) &\geq K_1 + \frac{2c^{p_{23}}}{h}\,(T_h-t)^{-\dfrac{p_{23}}{p_{33}-1}},\ t \in (t_0,T_h),\\ V_I'(t) &\geq K_1 + K_2\,(T_h-t)^{-\dfrac{p_{23}}{p_{33}-1}},\ t \in (t_0,T_h), \end{aligned}$$

with $K_2 = \dfrac{2c^{p_{23}}}{h}$.
Integrating this inequality from t_0 to T_h, we have

$$V_I(T_h) \geq V_I(t_0) + K_1\,(T_h-t_0) + K_2\int_{t_0}^{T_h} (T_h-t)^{-\dfrac{p_{23}}{p_{33}-1}}\,dt.$$

The boundedness of V_h requires $p_{23}+1 < p_{33}$, which contradicts $p_{23}+1 > p_{33}$.

Step 2. The upper blow-up rate estimate of V_h.
Since $p_{33} > 1$, of the remark 3.1, we have for $i = 1,\dots,I$

$$W_i(t) \leq C_{33}\,(T_h-t)^{-\dfrac{1}{p_{33}-1}}, \quad \forall t \in (0,T_h).$$

Also as V_h blows up in T_h, let I blowing up node. Then, there exists $A, C > 0$, $t_0 \in (0,T_h)$ such that

$$V_I'(t) \leq A V_I^{p_{22}}(t) + C\,(T_h-t)^{-\dfrac{p_{23}}{p_{33}-1}}, \quad \forall t \in (t_0,T_h),$$

integrating this inequality from t_0 to t, we obtain

$$V_I(t) \leq V_I(t_0) + AV_I^{p_{22}}(t)\,(t-t_0) - \frac{C\,(p_{33}-1)}{p_{33}-p_{23}-1}\,(T_h-t)^{\dfrac{p_{33}-p_{23}-1}{p_{33}-1}} + \frac{C\,(p_{33}-1)}{p_{33}-p_{23}-1}\,(T_h-t_0)^{\dfrac{p_{33}-p_{23}-1}{p_{33}-1}}, \quad \forall t \in (t_0, T_h),$$

which implies that

$$V_I(t) \leq V_I(t_0) + AV_I^{p_{22}}(t)\,(T_h-t_0) + \frac{C\,(p_{33}-1)}{p_{23}+1-p_{33}}\,(T_h-t)^{-\dfrac{p_{23}+1-p_{33}}{p_{33}-1}} - \frac{C\,(p_{33}-1)}{p_{23}+1-p_{33}}\,(T_h-t_0)^{-\dfrac{p_{23}+1-p_{33}}{p_{33}-1}}, \quad \forall t \in (t_0, T_h).$$

Since $p_{22} \leq 1$, then $V_I^{p_{22}}(t) \leq V_I(t)$.
Take t_0 such that $1 < V_I(t_0) < \dfrac{V_I(t)}{4}$ and $A(T_h-t_0) < \dfrac{1}{4}$, then, we obtain

$$V_I(t) \leq -\frac{2C\,(p_{33}-1)}{p_{23}+1-p_{33}}\,(T_h-t_0)^{-\dfrac{p_{23}+1-p_{33}}{p_{33}-1}} + \frac{2C\,(p_{33}-1)}{p_{23}+1-p_{33}}\,(T_h-t)^{-\dfrac{p_{23}+1-p_{33}}{p_{33}-1}}, \quad \forall t \in (t_0, T_h),$$

which implies that

$$V_I(t) \leq Q\,(T_h-t)^{-\dfrac{p_{23}+1-p_{33}}{(p_{33}-1)}}, \quad \forall t \in (t_0, T_h), \tag{4.1}$$

where $Q = \dfrac{2C\,(p_{33}-1)}{p_{23}+1-p_{33}}$.
Step 3. U_h remains bounded up to time T_h.
Suppose that U_h blows up et I blowing up node.
There exists $C', Q > 0$, $t_0 \in (0, T_h)$ such that from (4.1) and (1.2), we have

$$U_I'(t) \leq C_I'^{p_{11}}(t) + Q\,(T_h-t)^{-\dfrac{p_{12}\,(p_{23}+1-p_{33})}{p_{33}-1}}, \quad \forall t \in (t_0, T_h),$$

which implies that

$$\begin{aligned}
\frac{U_I'(t)}{U_I^{p_{11}}(t)} &\leq \frac{Q\,(T_h-t)^{-\dfrac{p_{12}\,(p_{23}+1-p_{33})}{p_{33}-1}}}{U_I^{p_{11}}(t)} + C', \quad \forall t \in (t_0, T_h),\\
\frac{U_I'(t)}{U_I^{p_{11}}(t)} &\leq Q\,(T_h-t)^{-\dfrac{p_{12}\,(p_{23}+1-p_{33})}{p_{33}-1}} + C', \quad \forall t \in (t_0, T_h)
\end{aligned}$$

because $(U_I(t) \geq 1)$.

Integrating this inequality from t_0 to t, we obtain, $\forall t \in (t_0, T_h)$

$$U_I(t) \leq \exp\left(\ln\left(U_I(t_0)\right) + Q_1 T_b^{\frac{p_{33}-1-p_{12}\left(p_{23}+1-p_{33}\right)}{p_{33}-1}} + C' T_b\right), \quad si \quad p_{11} = 1,$$

where $Q_1 = \dfrac{Q\left(p_{33}-1\right)}{p_{33}-1-p_{12}\left(p_{23}+1-p_{33}\right)}$, and $\forall t \in (t_0, T_h)$

$$U_I(t) \leq \left(U_I^{1-p_{11}}(t_0) + Q_2 T_b^{\frac{p_{33}-1-p_{12}\left(p_{23}+1-p_{33}\right)}{p_{33}-1}} + C_1' T_h\right)^{\frac{1}{1-p_{11}}}, \quad si \quad p_{11} < 1,$$

where $Q_2 = \left(1-p_{11}\right) Q_1$ et $C_1' = \left(1-p_{11}\right) C'$.
Which is a contradiction with U_h blows up and the proof is completed. $\square$

Theorem 4.5. *The case for only V_h remaining bounded up to T_h.*
If $1 < p_{11} < p_{31}+1$, $p_{22} \leq 1$, $p_{33} \leq 1$, and $p_{23} < \dfrac{p_{11}-1}{p_{31}+1-p_{11}}$, then, for every initial data, U_h and W_h blow up simultaneously at T_h while V_h remains bounded up to time T_h.

Theorem 4.6. *The case for only W_h remaining bounded up to T_h.*
If $1 < p_{22} < p_{12}+1$, $p_{11} \leq 1$, $p_{33} \leq 1$, and $p_{31} < \dfrac{p_{22}-1}{p_{12}+1-p_{22}}$, then, for every initial data, U_h and V_h blow up simultaneously at T_h while W_h remains bounded up to time T_h.

Theorems 4.5 and 4.6 prove themselves in the same way as Theorem 4.4.

5. Convergence of semidiscrete blow-up time

In this section, first under some assumptions, we show that the solution of the semidiscrete problem converges to the solution of the continuous problem. Then, we prove that the semediscrete blow-up time converges to the theoretical one when the mesh size tends to zero. Before, we denote

$$\begin{aligned} u_h(t) &= \left(u(x_1,t),\ldots,u(x_I,t)\right)^T, \quad v_h(t) = \left(v(x_1,t),\ldots,v(x_I,t)\right)^T, \\ w_h(t) &= \left(w(x_1,t),\ldots,w(x_I,t)\right)^T. \end{aligned}$$

Theorem 5.1. *Assume that the problem* (1) *has a solution $(u,v,w) \in \left(C^{4,1}\left([0,1]\times[0,T^*]\right)\right)^3$ and the initial data $\left(\psi_{1,h},\psi_{2,h},\psi_{3,h}\right)$ at* (1.5) *satisfies*

$$\|\psi_{1,h}-u_h(0)\|_\infty = o(1), \quad \|\psi_{2,h}-v_h(0)\|_\infty = o(1), \quad \|\psi_{3,h}-w_h(0)\|_\infty = o(1), \quad h \to 0 \tag{5.1}$$

Then, for h sufficiently small, the problem (1.2)–(1.5) *has a unique solution $(U_h, V_h, W_h) \in$*

$\left(C^1\left([0,T^*],\mathbb{R}^I\right)\right)^3$ *such that, as $h \to 0$*

$$\max_{t\in[0,T^*]}\|U_h(t)-u_h(t)\|_\infty = O(\|\psi_{1,h}-u_h(0)\|_\infty+\|\psi_{2,h}-v_h(0)\|_\infty+\|\psi_{3,h}-w_h(0)\|_\infty+h),$$
$$\max_{t\in[0,T^*]}\|V_h(t)-v_h(t)\|_\infty = O(\|\psi_{1,h}-u_h(0)\|_\infty+\|\psi_{2,h}-v_h(0)\|_\infty+\|\psi_{3,h}-w_h(0)\|_\infty+h),$$
$$\max_{t\in[0,T^*]}\|W_h(t)-w_h(t)\|_\infty = O(\|\psi_{1,h}-u_h(0)\|_\infty+\|\psi_{2,h}-v_h(0)\|_\infty+\|\psi_{3,h}-w_h(0)\|_\infty+h).$$

Proof. Let $\zeta > 0$ be such that

$$(\|u\|_\infty,\|v\|_\infty,\|w\|_\infty) < \zeta,\ \ t\in[0,T^*]. \tag{5.2}$$

Let $t(h) \leqslant T^*$ be the greatest value of $t > 0$ such that

$$\max\{\|U_h(t)-u_h(t)\|_\infty,\|V_h(t)-v_h(t)\|_\infty,\|W_h(t)-w_h(t)\|_\infty\} < 1, \quad t\in(0,t(h)). \tag{5.3}$$

The relation (5.1) implies $t(h) > 0$ for h small enough. Using the triangle inequality, we obtain

$$\|U_h(t)\|_\infty \leqslant 1+\zeta, \quad \|V_h(t)\|_\infty \leqslant 1+\zeta \text{ and } \|W_h(t)\|_\infty \leqslant 1+\zeta \text{ for } t\in(0,t(h)). \tag{5.4}$$

Let $(e_{1,i},e_{2,i},e_{3,i})(t) = (U_i-u_i,V_i-v_i,W_i-w_i)(t)$ for $i=1,\dots,I$, $\forall t\in[0,T^*]$ be the discretization error. These error functions verify

$$\begin{aligned}
e'_{1,i}(t) &= \delta^2 e_{1,i}(t) + p_{11}a_i(\varsigma_i(t))^{p_{11}-1}e_{1,i}(t) + p_{12}a_i(\lambda_i(t))^{p_{12}-1}e_{2,i}(t) + O(h),\\
e'_{2,i}(t) &= \delta^2 e_{2,i}(t) + p_{22}a_i(\lambda_i(t))^{p_{22}-1}e_{2,i}(t) + p_{23}a_i(\varrho_i(t))^{p_{23}-1}e_{3,i}(t) + O(h),\\
e'_{3,i}(t) &= \delta^2 e_{3,i}(t) + p_{33}a_i(\varrho_i(t))^{p_{33}-1}e_{3,i}(t) + p_{31}a_i(\varsigma_i(t))^{p_{31}-1}e_{1,i}(t) + O(h)
\end{aligned}$$

where $\varsigma_i(t)$, $\lambda_i(t)$ and $\varrho_i(t)$ lie, respectively, between $U_i(t)$ and $u(x_i,t)$, between $V_i(t)$ and $v(x_i,t)$ and between $W_i(t)$ and $w(x_i,t)$, for $i=1,\dots,I$. Using (5.2) and (5.4), there exist N_1 and N_2 positive constants such that

$$\begin{aligned}
e'_{1,i}(t) &\leqslant \delta^2 e_{1,i}(t) + a_iN_1|e_{1,i}(t)| + a_iN_1|e_{2,i}(t)| + N_2h, \quad i=1,\dots,I, \quad t\in[0,T^*]\\
e'_{2,i}(t) &\leqslant \delta^2 e_{2,i}(t) + a_iN_1|e_{2,i}(t)| + a_iN_1|e_{3,i}(t)| + N_2h, \quad i=1,\dots,I, \quad t\in[0,T^*]\\
e'_{3,i}(t) &\leqslant \delta^2 e_{3,i}(t) + a_iN_1|e_{3,i}(t)| + a_iN_1|e_{1,i}(t)| + N_2h, \quad i=1,\dots,I, \quad t\in[0,T^*].
\end{aligned}$$

Let $(f,g,q) \in \left(C^{4,1}([0,1],[0,T^*])\right)^3$ be such that
$f(x,t) = (\|\psi_{1,h}-u_h(0)\|_\infty+\|\psi_{2,h}-v_h(0)\|_\infty+\|\psi_{3,h}-w_h(0)\|_\infty+N_2h)\,\mathrm{e}^{(C_3+2)t+x^2-x}$ and $f=g=q$, $\forall(x,t)\in[0,1]\times[0,T^*]$, with C_3, N_2 positive constants. By the Lemma 2.2, we show that

$$(|e_{1,i}(t)|,|e_{2,i}(t)|,|e_{3,i}(t)|) < (f(x_i,t),g(x_i,t),q(x_i,t)) \text{ with } 1\leqslant i\leqslant I \text{ for } t\in(0,t(h)).$$

Thus we get

$$\begin{aligned}
&\|U_h(t)-u_h(t)\|_\infty \leqslant\\
&(\|\psi_{1,h}-u_h(0)\|_\infty+\|\psi_{2,h}-v_h(0)\|_\infty+\|\psi_{3,h}-w_h(0)\|_\infty+N_2h)\,\mathrm{e}^{(C_3+2)t},\\
&\|V_h(t)-v_h(t)\|_\infty \leqslant\\
&(\|\psi_{1,h}-u_h(0)\|_\infty+\|\psi_{2,h}-vh(0)\|_\infty+\|\psi_{3,h}-w_h(0)\|_\infty+N_2h)\,\mathrm{e}^{(C_3+2)t},\\
&\|W_h(t)-w_h(t)\|_\infty \leqslant\\
&(\|\psi_{1,h}-uh(0)\|_\infty+\|\psi_{2,h}-v_h(0)\|_\infty+\|\psi_{3,h}-w_h(0)\|_\infty+N_2h)\,\mathrm{e}^{(C_3+2)t},
\end{aligned}$$

where $t \in (0, t(h))$. Suppose that $t(h) < T^*$. From (5.3), we obtain

$$1 = \|U_h(t(h)) - u_h(t(h))\|_\infty \leqslant \\ (\|\psi_{1,h} - u_h(0)\|_\infty + \|\psi_{2,h} - v_h(0)\|_\infty + \|\psi_{3,h} - w_h(0)\|_\infty + N_2 h)\, \mathrm{e}^{(C_3+2)t(h)}.$$

Since the term on the right hand side of the above inequality goes to zero as h tends to zero, we deduce that $1 \leqslant 0$, which is impossible. Consequently $t(h) = T^*$ and we conclude the proof. □

Theorem 5.2. *Suppose that the problem* (1.1) *has a solution* (u, v, w) *which blows up in a finite time* T *such that* $(u, v, w) \in \left(C^{4,1}([0,1] \times [0,T))\right)^3$ *and the initial data at* (1.5) *satisfies* (5.1). *Under the assumption of theorem 3.2, the problem* (1.2)–(1.5) *has a solution* (U_h, V_h, W_h) *which blows up in a finite time* T_h *and we have*

$$\lim_{h \to 0} T_h = T.$$

Proof. Set $\sigma > 0$, there exists $\gamma > 0$ such that

$$\frac{y^{1-p_{11}}}{\alpha(p_{11}-1)} \leqslant \frac{\sigma}{2}, \quad \gamma \leqslant y. \tag{5.5}$$

Since u blows up in a finite time T, there exists a time $T_1 \in (T - \sigma/2; T)$ such that $\|u(\cdot,t)\|_\infty \geq 2\gamma$ for $t \in [T_1, T)$. Denote $T_2 = \dfrac{T_1 + T}{2}$, we see easily that $\sup_{t\in[0,T_2]} \|u(\cdot,t)\|_\infty < \infty$. It follows from Theorem 5.1 that for h sufficiently small

$$\sup_{t \in [0,T_2]} \|U_h(t) - u_h(t)\|_\infty \leqslant \gamma.$$

Applying the triangle inequality, we get

$$\|U_h(T_2)\|_\infty \geq \|u_h(T_2)\|_\infty - \|U_h(T_2) - u_h(T_2)\|_\infty \geq \gamma.$$

From Theorem 3.2, U_h blows up at the time T_h . We deduce from Remark 3.1 and (5.5) that

$$|T_h - T| \leqslant |T_h - T_2| + |T_2 - T| \leqslant \frac{\|U_h(T_2)\|_\infty^{1-p_{11}}}{\alpha(p_{11}-1)} + \frac{\sigma}{2} \leqslant \sigma.$$

Cases where V_h, W_h blows up is analogous. □

6. Numerical experiments

In this part, we present some numerical approximations to the non-simultaneous blow-up time of (1.1). We consider the following explicit scheme

$$
\begin{aligned}
U_1^{(n+1)} &= \left(1-\frac{2\Delta t_n}{h^2}\right)U_1^{(n)}+\frac{2\Delta t_n}{h^2}U_2^{(n)}+\frac{2\Delta t_n}{h}\left((U_1^{(n)})^{p_{11}}+(V_1^{(n)})^{p_{12}}\right),\\
U_i^{(n+1)} &= \frac{\Delta t_n}{h^2}U_{i-1}^{(n+1)}+\left(1-\frac{2\Delta t_n}{h^2}\right)U_i^{(n)}+\frac{\Delta t_n}{h^2}U_{i+1}^{(n+1)},\quad 2\le i\le I-1\\
U_I^{(n+1)} &= \frac{2\Delta t_n}{h^2}U_{I-1}^{(n)}+\left(1-\frac{2\Delta t_n}{h^2}\right)U_I^{(n)}+\frac{2\Delta t_n}{h}\left((U_I^{(n)})^{p_{11}}+(V_I^{(n)})^{p_{12}}\right),\\
V_1^{(n+1)} &= \left(1-\frac{2\Delta t_n}{h^2}\right)V_1^{(n)}+\frac{2\Delta t_n}{h^2}V_1^{(n)}+\frac{2\Delta t_n}{h}\left((V_1^{(n)})^{p_{22}}+(W_1^{(n)})^{p_{23}}\right),\\
V_i^{(n+1)} &= \frac{\Delta t_n}{h^2}V_{i-1}^{(n+1)}+\left(1-\frac{2\Delta t_n}{h^2}\right)V_i^{(n)}+\frac{\Delta t_n}{h^2}V_{i+1}^{(n+1)},\quad 2\le i\le I-1\\
V_I^{(n+1)} &= \frac{2\Delta t_n}{h^2}V_{I-1}^{(n)}+\left(1-\frac{2\Delta t_n}{h^2}\right)V_I^{(n)}+\frac{2\Delta t_n}{h}\left((V_I^{(n)})^{p_{22}}+(W_I^{(n)})^{p_{23}}\right),\\
W_1^{(n+1)} &= \left(1-\frac{2\Delta t_n}{h^2}\right)W_1^{(n)}+\frac{2\Delta t_n}{h^2}W_2^{(n)}+\frac{2\Delta t_n}{h}\left((W_1^{(n)})^{p_{33}}+(U_1^{(n)})^{p_{31}}\right),\\
W_i^{(n+1)} &= \frac{\Delta t_n}{h^2}W_{i-1}^{(n+1)}+\left(1-\frac{2\Delta t_n}{h^2}\right)W_i^{(n)}+\frac{\Delta t_n}{h^2}W_{i+1}^{(n+1)},\quad 2\le i\le I-1\\
W_I^{(n+1)} &= \frac{2\Delta t_n}{h^2}W_{I-1}^{(n)}+\left(1-\frac{2\Delta t_n}{h^2}\right)W_I^{(n)}+\frac{2\Delta t_n}{h}\left((W_I^{(n)})^{p_{33}}+(U_I^{(n)})^{p_{31}}\right),\\
&\qquad U_i^{(0)}=\psi_{1,i},\quad V_i^{(0)}=\psi_{2,i},\quad W_i^{(0)}=\psi_{3,i},\quad 1\le i\le I,
\end{aligned}
$$

where $n\ge 0,\ p_{11},p_{22},p_{33},p_{12},p_{23},p_{31}\ge 0$,

$$\Delta t_{n,1} =$$

$$\frac{\tau h}{2}\min\left\{\|U_h^{(n)}\|_\infty^{1-p_{11}},\|V_h^{(n)}\|_\infty^{1-p_{12}},\|V_h^{(n)}\|_\infty^{1-p_{22}},\|W_h^{(n)}\|_\infty^{1-p_{23}},\|W_h^{(n)}\|_\infty^{1-p_{33}},\|U_h^{(n)}\|_\infty^{1-p_{31}}\right\},$$

$\Delta t_n=\min\left\{\frac{h^2}{2},\Delta t_{n,1}\right\}$ for $0<\tau<1$.

We also consider the implicit scheme

$$
\begin{aligned}
&\left(1+\frac{2\Delta t_n}{h^2}\right)U_1^{(n+1)}-\frac{2\Delta t_n}{h^2}U_2^{(n+1)}=U_1^{(n)}+\frac{2\Delta t_n}{h}\left((U_1^{(n)})^{p_{11}}+(V_1^{(n)})^{p_{12}}\right),\\
&(-\frac{\Delta t_n}{h^2})U_{i-1}^{(n+1)}+\left(1+\frac{2\Delta t_n}{h^2}\right)U_i^{(n+1)}-(\frac{\Delta t_n}{h})U_{i+1}^{(n+1)}=U_i^{(n)},\quad 2\le i\le I-1\\
&(-\frac{\Delta t_n}{h^2})U_{I-1}^{(n+1)}+\left(1+\frac{2\Delta t_n}{h^2}\right)U_I^{(n+1)}=U_I^{(n)}+\frac{2\Delta t_n}{h}\left((U_I^{(n)})^{p_{11}}+(V_I^{(n)})^{p_{12}}\right),
\end{aligned}
$$

$$\left(1+\frac{2\Delta t_n}{h^2}\right)V_1^{(n+1)}-\frac{2\Delta t_n}{h^2}V_2^{(n+1)}=V_1^{(n)}+\frac{2\Delta t_n}{h}\left((V_1^{(n)})^{p_{22}}+(W_1^{(n)})^{p_{23}}\right),$$
$$(-\frac{\Delta t_n}{h^2})V_{i-1}^{(n+1)}+\left(1+\frac{2\Delta t_n}{h^2}\right)V_i^{(n+1)}-(\frac{\Delta t_n}{h})V_{i+1}^{(n+1)}=V_i^{(n)},\quad 2\le i\le I-1$$
$$(-\frac{\Delta t_n}{h^2})V_{I-1}^{(n+1)}+\left(1+\frac{2\Delta t_n}{h^2}\right)V_I^{(n+1)}=V_I^{(n)}+\frac{2\Delta t_n}{h}\left((V_I^{(n)})^{p_{22}}+(W_I^{(n)})^{p_{23}}\right),$$
$$\left(1+\frac{2\Delta t_n}{h^2}\right)W_1^{(n+1)}-\frac{2\Delta t_n}{h^2}W_2^{(n+1)}=W_1^{(n)}+\frac{2\Delta t_n}{h}\left((W_1^{(n)})^{p_{33}}+(U_1^{(n)})^{p_{31}}\right),$$
$$(-\frac{\Delta t_n}{h^2})W_{i-1}^{(n+1)}+\left(1+\frac{2\Delta t_n}{h^2}\right)W_i^{(n+1)}-(\frac{\Delta t_n}{h})W_{i+1}^{(n+1)}=W_i^{(n)},\quad 2\le i\le I-1$$
$$(-\frac{\Delta t_n}{h^2})W_{I-1}^{(n+1)}+\left(1+\frac{2\Delta t_n}{h^2}\right)W_I^{(n+1)}=W_I^{(n)}+\frac{2\Delta t_n}{h}\left((W_I^{(n)})^{p_{33}}+(U_I^{(n)})^{p_{31}}\right),$$
$$U_i^{(0)}=\psi_{1,i},\quad V_i^{(0)}=\psi_{2,i},\quad W_i^{(0)}=\psi_{3,i},\quad 1\le i\le I,$$

where $n\ge 0,\ p_{11},p_{22},p_{33},p_{12},p_{23},p_{31}\ge 0$,

$$\Delta t_n=$$
$$\frac{\tau h}{2}\min\left\{\|U_h^{(n)}\|_\infty^{1-p_{11}},\|V_h^{(n)}\|_\infty^{1-p_{12}},\|V_h^{(n)}\|_\infty^{1-p_{22}},\|W_h^{(n)}\|_\infty^{1-p_{23}},\|W_h^{(n)}\|_\infty^{1-p_{33}},\|U_h^{(n)}\|_\infty^{1-p_{31}}\right\}$$

for $0<\tau<1$.

In both cases, we use : $\tau=h$, $\psi_{1,i}=\psi_{2,i}=\psi_{3,i}=((i-1)h)^2$, $i=1,\dots,I$.

In the tables 1-12, in rows, we present the numerical non-simultaneous blow-up times, numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256, 512. The numerical non-simultaneous blow-up time $T^n=\sum_{j=0}^{n-1}\Delta t_j$ is computed at the first time when $\Delta t_n=|T^{n+1}-T^n|\le 10^{-16}$. The order(s) of the method is computed from

$$s=\frac{\log((T_{4h}-T_{2h})/(T_{2h}-T_h))}{\log(2)}.$$

First case : $(p_{11},p_{22},p_{33},p_{12},p_{23},p_{31})=(4;1/2;1/4;4;6;2)$.

Table 1. Explicit Euler method

I	T^n	n	$CPUt$	s
16	0.041898128	200	0.05	-
32	0.037901493	452	0.09	-
64	0.036665418	1172	0.38	1.69
128	0.036296227	3513	2.08	1.74
256	0.036188702	11891	13.73	1.78
512	0.036157985	43602	108.88	1.81

Table 2. Implicit Euler method

I	T^n	n	$CPUt$	s
16	0.044008703	202	0.06	-
32	0.038556888	455	0.09	-
64	0.036845859	1176	3.34	1.67
128	0.036343378	3418	25.02	1.77
256	0.036200741	11897	242.02	1.82
512	0.036161026	43610	5634.52	1.84

Second case : $(p_{11}, p_{22}, p_{33}, p_{12}, p_{23}, p_{31}) = (1/2; 4; 1/4; 2; 2; 2)$.

TABLE 3. Explicit Euler method

I	T^n	n	$CPUt$	s
16	0.054001945	213	0.05	-
32	0.049910667	501	0.13	-
64	0.048644971	1361	0.44	1.69
128	0.048267027	4264	2.53	1.74
256	0.048157102	14886	17.06	1.78
512	0.048125759	55577	124.53	1.81

TABLE 4. Implicit Euler method

I	T^n	n	$CPUt$	s
16	0.056116881	215	0.08	-
32	0.050553087	504	0.14	-
64	0.048821084	1365	4.61	1.68
128	0.048313043	4269	28.50	1.77
256	0.048168855	14893	296.22	1.81
512	0.048128728	55585	7219.93	1.85

Third case : $(p_{11}, p_{22}, p_{33}, p_{12}, p_{23}, p_{31}) = (1/4; 1/2; 4; 4; 2; 4)$.

TABLE 5. Explicit Euler method

I	T^n	n	$CPUt$	s
16	0.044858762	203	0.03	-
32	0.040878827	466	0.09	-
64	0.039646397	1226	0.42	1.69
128	0.039277697	3726	2.36	1.74
256	0.039170208	12741	14.73	1.78
512	0.039139487	47001	109.63	1.81

TABLE 6. Implicit Euler method

I	T^n	n	$CPUt$	s
16	0.046934645	206	0.05	-
32	0.041516476	469	0.11	-
64	0.039821621	1230	3.40	1.68
128	0.039323497	3732	24.72	1.77
256	0.039181907	12747	263.20	1.81
512	0.039142443	47008	5817.09	1.84

Fourth case : $(p_{11}, p_{22}, p_{33}, p_{12}, p_{23}, p_{31}) = (1/2; 2; 1/4; 2; 4; 1/2)$.

TABLE 7. Explicit Euler method

I	T^n	n	$CPUt$	s
16	0.152103798	656	0.11	-
32	0.146438502	1606	0.28	-
64	0.144697707	4561	1.39	1.70
128	0.144180521	14826	8.69	1.75
256	0.144030644	53035	61.02	1.79
512	0.143988023	200702	447.20	1.81

TABLE 8. Implicit Euler method

I	T^n	n	$CPUt$	s
16	0.154375956	660	0.11	-
32	0.147088116	1611	0.33	-
64	0.144870842	4568	16.31	1.72
128	0.144225200	14835	101.13	1.78
256	0.144041998	53045	1067.52	1.82
512	0.143990887	200713	23534.97	1.84

Fifth case : $(p_{11}, p_{22}, p_{33}, p_{12}, p_{23}, p_{31}) = (4; 1/4; 1/10; 4; 1/4; 4)$.

Table 9. Explicit Euler method

I	T^n	n	$CPUt$	s
16	0.050491301	210	0.05	-
32	0.046596522	491	0.09	-
64	0.045382351	1322	0.42	1.68
128	0.045017496	4108	2.63	1.73
256	0.044910860	14267	17.27	1.77
512	0.044880339	53101	120.88	1.80

Table 10. Implicit Euler method

I	T^n	n	$CPUt$	s
16	0.052435320	212	0.08	-
32	0.047189977	494	0.13	-
64	0.045545720	1326	3.36	1.67
128	0.045060291	4114	26.73	1.76
256	0.044921805	14273	279.23	1.81
512	0.044883107	53109	6306.59	1.84

Sixth case : $(p_{11}, p_{22}, p_{33}, p_{12}, p_{23}, p_{31}) = (1/4; 1/2; 4; 1; 4; 4)$.

Table 11. Explicit Euler method

I	T^n	n	$CPUt$	s
16	0.048715903	208	0.05	-
32	0.044733875	482	0.13	-
64	0.043497195	1288	0.41	1.69
128	0.043126812	3975	2.28	1.74
256	0.043018821	13735	15.67	1.78
512	0.042987966	50973	115.31	1.81

Table 12. Implicit Euler method

I	T^n	n	$CPUt$	s
16	0.050750286	210	0.08	-
32	0.045357772	485	0.13	-
64	0.043668819	1293	3.36	1.67
128	0.043171711	3981	27.91	1.76
256	0.043030295	13741	314.56	1.81
512	0.042990866	50981	6439.55	1.84

In the following, we also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes in the case where $I = 32$.

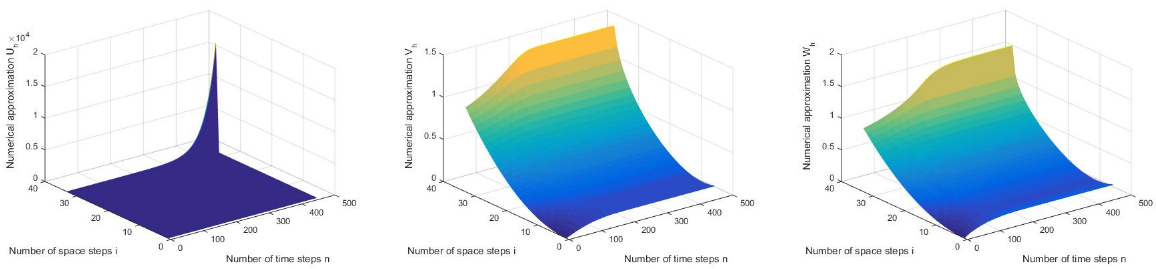

Figure 1. (Explicit scheme): The component U_h blows up and the components V_h and W_h are bounded.

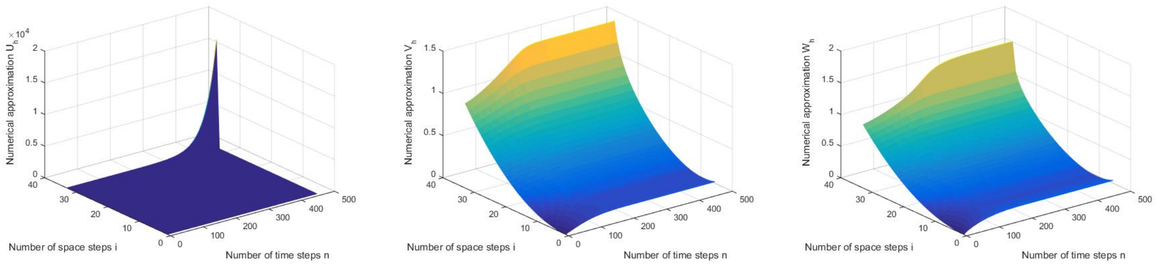

FIGURE 2. (Implicit scheme): The component U_h blows up and the components V_h and W_h are bounded.

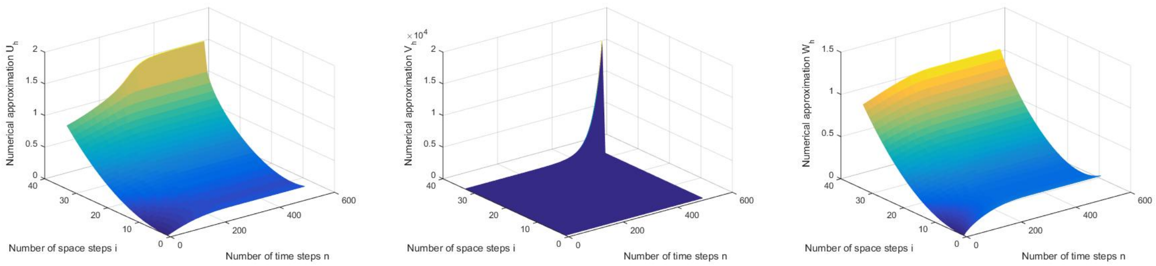

FIGURE 3. (Explicit scheme): The component V_h blows up and the components U_h and W_h are bounded.

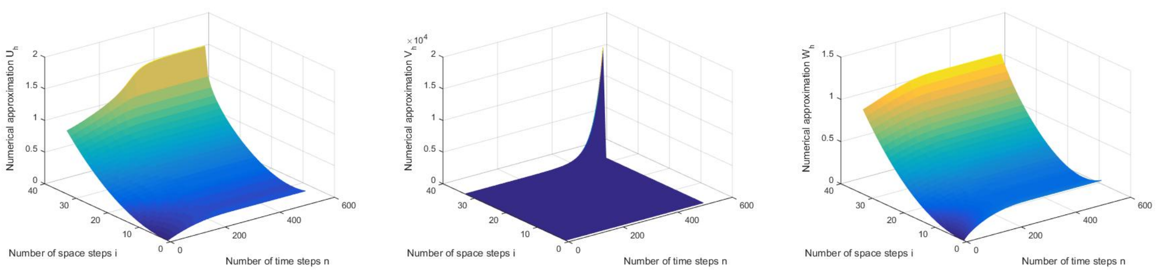

FIGURE 4. (Implicit scheme): The component V_h blows up and the components U_h and W_h are bounded.

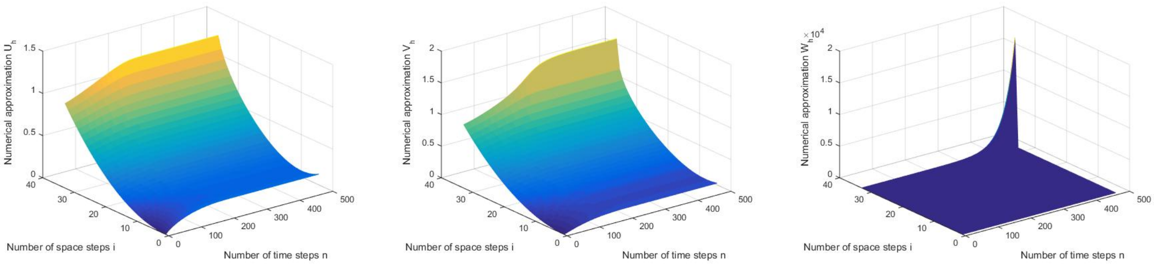

FIGURE 5. (Explicit scheme): The component W_h blows up and the components U_h and V_h are bounded.

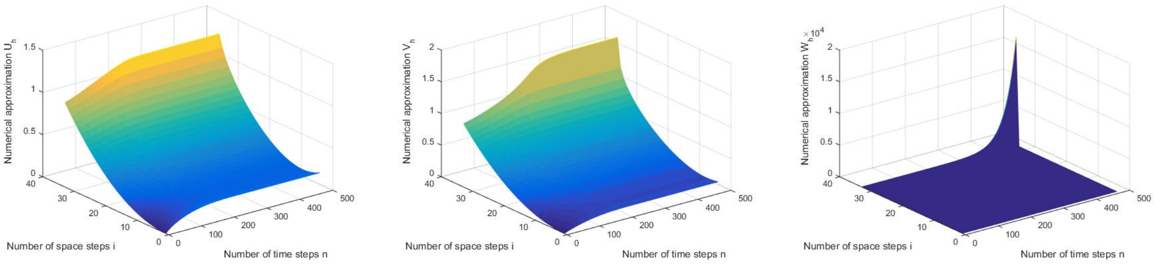

FIGURE 6. (Implicit scheme): The component W_h blows up and the components U_h and V_h are bounded.

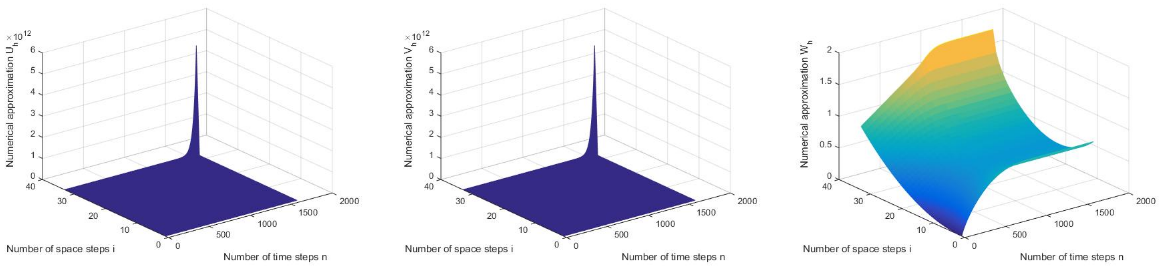

FIGURE 7. (Explicit scheme): The components U_h and V_h blow up simultaneously and the component W_h is bounded.

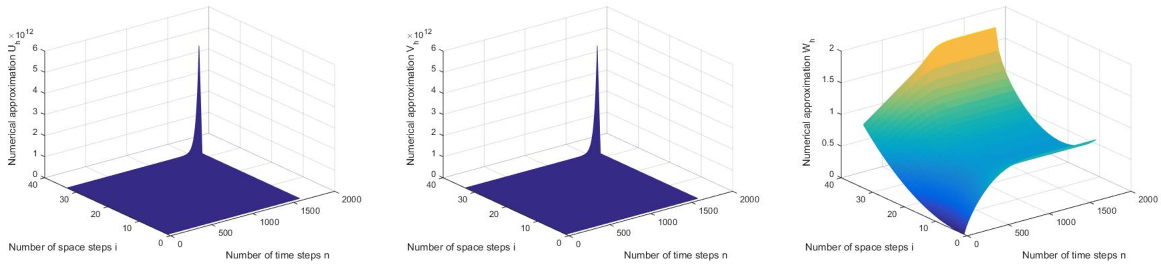

FIGURE 8. (Implicit scheme): The components U_h and V_h blow up simultaneously and the component W_h is bounded.

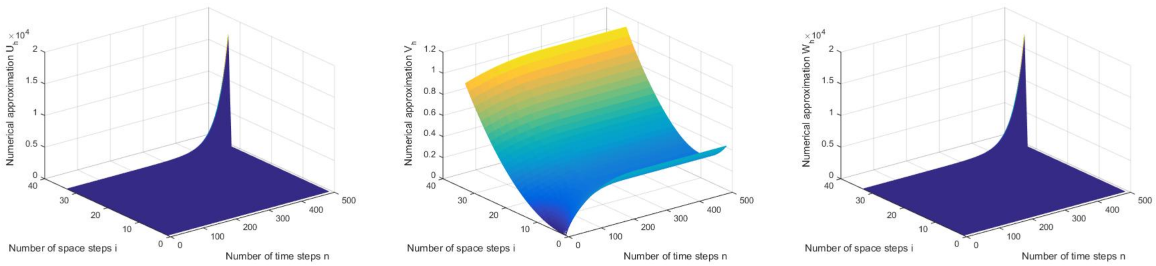

FIGURE 9. (Explicit scheme): The component U_h and W_h blow up simultaneously and the component V_h is bounded.

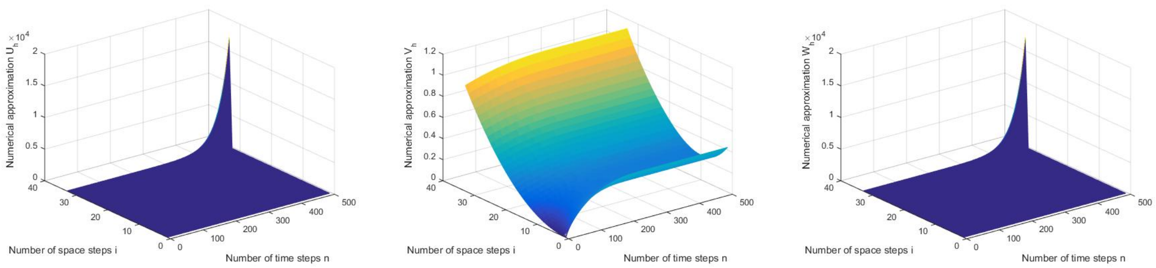

FIGURE 10. (Implicit scheme): The component U_h and W_h blow up simultaneously and the component V_h is bounded.

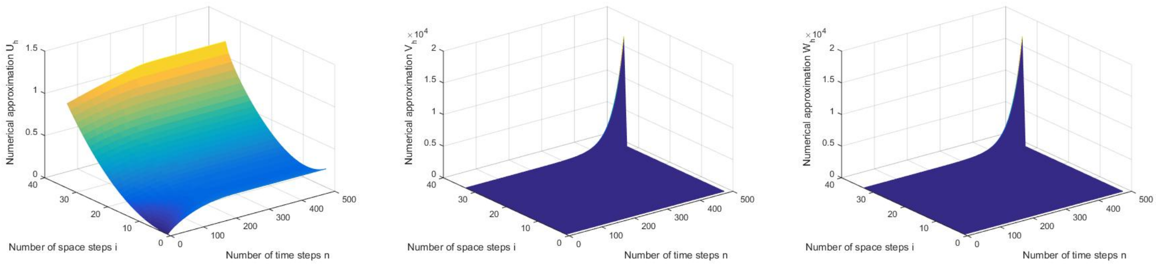

FIGURE 11. (Explicit scheme): The components V_h and W_h blow up simultaneously and the component U_h is bounded.

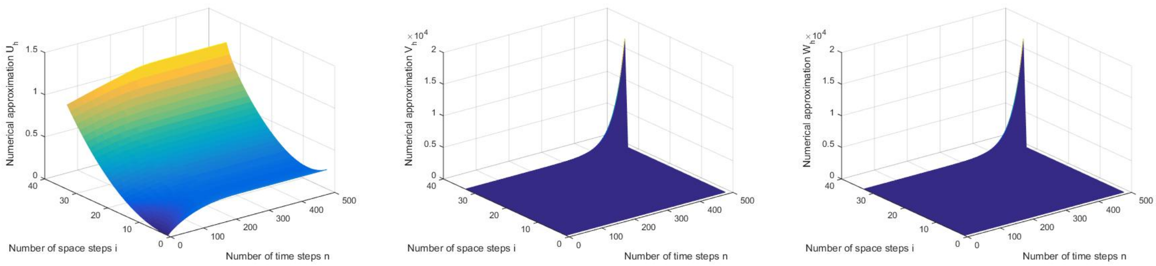

FIGURE 12. (Implicit scheme): The components V_h and W_h blow up simultaneously and the component U_h is bounded.

7. Main results

In fact, when:
$p_{22} \leq 1$, $p_{33} \leq 1$ and $p_{31} + 1 < p_{11}$ only U_h blows up for every positive initial data.
$p_{11} \leq 1$, $p_{33} \leq 1$ and $p_{12} + 1 < p_{22}$ only V_h blows up for every positive initial data.
$p_{11} \leq 1$, $p_{22} \leq 1$ and $p_{23} + 1 < p_{33}$ only W_h blows up for every positive initial data.
$1 < p_{22} < p_{12} + 1$, $p_{11} \leq 1$, $p_{33} \leq 1$, and $p_{31} < \dfrac{p_{22} - 1}{p_{12} + 1 - p_{22}}$) for every initial data, U_h and V_h blow up simultaneously at T_h while W_h remains bounded up to time T_h.
$1 < p_{11} < p_{31} + 1$, $p_{22} \leq 1$, $p_{33} \leq 1$, and $p_{23} < \dfrac{p_{11} - 1}{p_{31} + 1 - p_{11}}$) for every initial data, U_h and W_h blow up simultaneously at T_h while V_h remains bounded up to time T_h.
$1 < p_{33} < p_{23} + 1$, $p_{11} \leq 1$, $p_{22} \leq 1$, and $p_{12} < \dfrac{p_{33} - 1}{p_{23} + 1 - p_{33}}$ for every initial data, V_h and W_h blow up simultaneously at T_h while U_h remains bounded up to time T_h.
We remark that the non-smultaneous blow-up occurs, which confirms the results known theoretical of the problem (1.1) (see [3]).
For the tables 1 and 2 with $(p_{11}; p_{12}; p_{22}; p_{23}; p_{33}; p_{31}) = (4; 1/2; 1/4; 4; 6; 2)$, an approximate value of the simultaneous blow-up time is 0.03.
For the tables 3 and 4 with $(p_{11}; p_{12}; p_{22}; p_{23}; p_{33}; p_{31}) = (4; 1/4; 1/10; 4; 1/4; 4)$ an approximate value of the simultaneous blow-up time is 0.04.

For the tables 5 and 6 with $(p_{11}; p_{12}; p_{22}; p_{23}; p_{33}; p_{31}) = (1/4; 1/2; 4; 1; 4; 4)$ an approximate value of the simultaneous blow-up time is 0.03.
For the tables 7 and 8 with $(p_{11}; p_{12}; p_{22}; p_{23}; p_{33}; p_{31}) = (1/2; 2; 1/4; 2; 4; 1/2)$, an approximate value of the simultaneous blow-up time is 0.1.
For the tables 9 and 10 with $(p_{11}; p_{12}; p_{22}; p_{23}; p_{33}; p_{31}) = (4; 1/4; 1/10; 4; 1/4; 4)$ an approximate value of the simultaneous blow-up time is 0.04.
For the tables 11 and 12 with $(p_{11}; p_{12}; p_{22}; p_{23}; p_{33}; p_{31}) = (1/4; 1/2; 4; 1; 4; 4)$ an approximate value of the simultaneous blow-up time is 0.04.

8. Conclusion

In this paper, we studied a numerical approximation of a reaction-diffusion system with three components. First, we have constructed a semidiscrete scheme of the continuous problem and we have have given some properties of this semidiscrete scheme such as the blow-up of the semidiscrete solution and we have estimated its semidiscrete blow-up time. Then, under certain conditions, we have proved that the semidiscrete non-simultaneous blow-up occurs. In addition, we have established the convergence of the semidiscrete blow-up time to the theoretical one when the mesh size tends to zero. Finally, we have given some numerical experiments to illustrate our analysis.

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[1] Université Virtuelle de Côte d'Ivoire, 28 BP 536 Abidjan 28, (Côte d'Ivoire)
[2] Institut National Polytechnique Houphouët-Boigny de Yamoussoukro, BP 1093 Yamoussoukro, (Côte d'Ivoire)
[3] Université Alassane Ouattara de Bouaké, UFR-SEG, 01 BP V 18 Bouaké 01, (Côte d'Ivoire)
[4] Institut National Polytechnique Houphouët-Boigny de Yamoussoukro, BP 2444 Yamoussoukro, (Côte d'Ivoire)
Email address: [1] berengeredja@gmail.com, [2] mathieulouangemarieelie@gmail.com, [3] nkrasoft@yahoo.fr, [4] latoureci@gmail.com